Comparative Study of *A*-topology with Other Topologies

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ABSTRACT

Minkowski space, M, is a mathematical model for space-time of special relativity theory. There are various topologies on M which reflect desirable physical properties. Such physically significant topologies on M include t-topology, s-topology, fine-topology, space-topology, A-topology etc. The A-topology on M is defined to be the finest topology on M with respect to which the induced topology on every time-like line and light-like line is one-dimensional Euclidean and the induced topology on every space-like hyper-plane is three-dimensional Euclidean. In the present paper, a characterization of open sets and closed sets of M^A has been found. Several examples have been worked out. Further, It has been obtained that A-topology is coarser than time, space and fine topology, but not comparable with t, s and f-topology.

Keywords: Minkowski space, A-topology, time-like line, light-like line, space-like hyper-plane, open sets and closed sets of M^A.

1. INTRODUCTION

Einstein introduced ‘Theory of Special Relativity’ in flat space-time which was compatible with classical electromagnetism and developed approximate theory of space-time while gravity and acceleration of observer were ignored. German mathematician Minkowski realized that Einstein’s theory of Special Relativity can be appreciated in a space-time defined by using Lorentz inner product. This four dimensional space-time after Minkowski was named as “Minkowski Space”. The key point of Minkowski’s idea is that the geometry of four-dimensional space-time does not separate out naturally into a time dimension and a family of ordinary Euclidean 3-spaces. Minkowski space-time has a different kind of geometric structure, giving a curious twist to Euclid’s ancient idea of geometry. It provides an overall geometry to space-time, making it one indivisible whole, which completely encodes the structure of Einstein’s special relativity. Minkowski’s space-time continuum is flat and non-curved because of absence of gravity.

The most natural topology on *M* is Euclidean topology, generated by Euclidean distance function. It is not appropriate choice for the topology on *M* because (i) Euclidean topology is homogeneous, whereas *M* is not because light cone associated with it separates space-like and time-like vectors (ii) its homeomorphism group is very large and of no physical significance. Fine topology was first non-Euclidean topology on M. The fine topology on Minkowski space encodes the information of the causal and linear structure of Minkowski space. The fine topology encodes much interesting information about the original space-time, as it induces discrete topology on light ray. Intuitively this shows that the track of a photon is not a continuous path. This implies that photons are excluded from the category of particles whose paths are continuous. If, however, one wants to include the photons in this category, then the fine topology will be unsuitable and we have to put a new topology on M known as A-topology, defined by Nanda [6].

The present paper is focused on the comparative study of A-topology with other non-Euclidean topologies on M. The paper begins with the necessary notation and preliminaries in Section 2. Open and closed subsets of M^A have been studied in Section 3. In Section 4, comparison of A-topology with other non-Euclidean topologies on M has been studied. Finally, Section 5 has the conclusion of research work.

2. NOTATION AND PRELIMINARIES

Throughout R, Q, M, ρ, τ and μ denote set of real numbers, set of rational numbers, n-dimensional Minkowski space, space like hyper-plane, time-like line and light-like line respectively. The spaces M^E, M^F, M^f, M^t, M^s and M^A denote Minkowski space with Euclidean topology, fine-topology, f-topology, t-topology, s-topology and A-topology respectively.

The n-dimensional real vector space R^n with bilinear form g: $R^n \times R^n \to R$, satisfying the following properties: (i) symmetric, i.e. $g(x, y) = g(y, x)$ for all $x, y \in R^n$ (ii) non-degenerate, i.e.

$g(x, y) = 0$, then $x = 0$ for all x and $y \in R^n$ and (iii) there exists a basis $e_0, e_1, \ldots, e_{n-1}$ for R^n with $g(e_i, e_j) = \eta_{ij} = \begin{cases} 1, \text{ if } i = j = 0 \\ -1, \text{ if } i = j = 1, \ldots, n-1 \\ 0, \text{ if } i \neq j \end{cases}$ is called the *n-dimensional Minkowski space*. The bilinear form g is called the *Lorentz inner product* and the matrix η_{ij} is known as the *Minkowski metric*. An element $x \in M$ is called *event*. For $x = \sum_{i=0}^{n-1} x_i e_i$, the coordinate x_0 is called the time component and the coordinates $x_1, x_2, \ldots, x_{n-1}$ are called the spatial components of x relative to the basis $e_0, e_1, \ldots, e_{n-1}$. Lorentz inner product of two events x and y is denoted by $g(x, y)$ and is equal to $x_0 y_0 - \sum_{i=1}^{n-1} x_i y_i$, where $x = \sum_{i=0}^{n-1} x_i e_i$ and $y = \sum_{i=0}^{n-1} y_i e_i$. An indefinite quadratic form Q on M induced by Lorentz inner product is defined as $Q(x) = g(x, x)$, for all $x \in M$. The group of all linear operators T on M which leave the quadratic form Q invariant, i.e., $Q(x) = Q(Tx)$ for all $x \in M$, is called the *Lorentz group*. An event $x \in M$ is called time-like, light-like or null or space-like, accordingly as $Q(x)$ is positive, zero, or negative. For $x \in M$, the three sets $C^T(x) = \{y \in M : Q(y - x) > 0\} \cup \{x\}$, $C^L(x) = \{y \in M: Q(y - x) = 0\}$ and $C^S(x) = \{y \in M: Q(y - x) < 0\} \cup \{x\}$ are called the *time cone*, *light cone* or *null cone*, and *space cone* respectively at x. A straight line is called a *time-like straight line* or *light ray* or *space-like straight line,* accordingly, as it is parallel to a time-like or light-like or space-like vector.

3. SUBSETS OF M^A

In this section, a necessary and sufficient condition for a set to be open in M with A-topology, i.e. M^A, has been obtained. Also a characterization of closed set of M^A has been found. Several examples have been worked out.

3.1 Definition

The *A-topology* on four-dimensional Minkowski space M is defined to be the finest topology on M with respect to which the induced topology on every time-like line and light-like line is one-dimensional Euclidean and the induced topology on every space-like hyper-plane is three-dimensional Euclidean [6]. In a similar manner, A-topology on n-dimensional Minkowski space can be defined. It is finer than the Euclidean topology and hence it is Hausdorff.

Proposition 3.2: Let M^A be the n-dimensional Minkowski space with A-topology and G be a non-empty subset of M. Then G is open in M^A if and only if $G \cap \rho$, $G \cap \tau$ and $G \cap \mu$ are open in ρ^E, τ^E and μ^E respectively.

Proof: If G is open in M^A then by the definition of A-topology on M, $G \cap \rho$, $G \cap \tau$ and $G \cap \mu$ are open in ρ^E, τ^E and μ^E respectively. Conversely, let T be the topology generated by the basis $B = \{G \subseteq M: G \cap \rho, G \cap \tau$ and $G \cap \mu$ are open in ρ^E, τ^E and μ^E respectively$\}$. Clearly A-topology is coarser than T. Let $H \in T$. Then $H \cap \rho$, $H \cap \tau$ and $H \cap \mu$ are open in ρ^E, τ^E and μ^E respectively, because H is a union of element of B. By definition, A-topology is the finest such topology. Hence topology $T = A$. This shows that G is open in A-topology.

Proposition 3.3: *Let M^A be the n-dimensional Minkowski space with A-topology and F be a non-empty subset of M. Then F is closed in M^A if and only if $F \cap \rho$, $F \cap \tau$ and $F \cap \mu$ are closed in ρ^E, τ^E and μ^E respectively.*

Proof: Let F be closed in M^A. Then $M - F$ is open in M^A, $(M - F) \cap \rho$, $(M - F) \cap \tau$ and $(M - F) \cap \mu$ are open in ρ^E, τ^E and μ^E respectively. This implies that $\{\rho - (F \cap \rho)\}$, $\{\tau - (F \cap \tau)\}$ and $\{\mu - (F \cap \mu)\}$ are open in ρ^E, τ^E and μ^E respectively. Hence $(F \cap \rho)$, $(F \cap \tau)$ and $(F \cap \mu)$ are closed in ρ^E, τ^E and μ^E respectively.

Conversely, let $(F \cap \rho)$, $(F \cap \tau)$ and $(F \cap \mu)$ are closed in ρ^E, τ^E and μ^E respectively. This implies that $\{\rho - (F \cap \rho)\}$, $\{\tau - (F \cap \tau)\}$ and $\{\mu - (F \cap \mu)\}$ are open in ρ^E, τ^E and μ^E respectively. Further,

$(M - F) \cap \rho$, $(M - F) \cap \tau$ and $(M - F) \cap \mu$ are open in ρ^E, τ^E and μ^E respectively. This implies that $(M - F)$ is open in M^A, hence F is closed in M^A.

Proposition 3.4: *Let M^A be the n-dimensional Minkowski space with A-topology. Then* $C^T(0)$ - $\{0\}$ *and* $C^S(0)$ - $\{0\}$ *are open in* M^A.

Proof: We have proved only that $(C^T(0)$ - $\{0\})$ is open in M^A. Since $(C^T(0)$ - $\{0\})$ is open in M^E and A-topology is finer than Euclidean topology [6], hence $(C^T(0)$ - $\{0\})$ is open in M^A.

Proposition 3.5: *Let M^A be the n-dimensional Minkowski space with A-topology. Then singletons are not open in M^A.*

Proof: Let $x \in M$ then for any $\varepsilon > 0$, there exist no open ball $N_\varepsilon^E(x)$ such that $N_\varepsilon^E(x) \subseteq \{x\}$ because $N_\varepsilon^E(x)$ has infinitely many points. Hence $\{x\}$ is not open in ρ^E. And by Proposition 3.2, $\{x\}$ is not open in M^A.

Proposition 3.6: *Let M^A be the n-dimensional Minkowski space with A-topology. Then* $C^L(0)$, $C^T(0)$ *and* $C^S(0)$ *are not open in* M^A.

Proof: We have proved only that $C^L(0)$ is not open in M^A. Let ρ be the space-like hyper-plane passing through origin respectively. Since $C^L(0) \cap \rho = \{0\}$ and singletons are not open in ρ^E. This implies that $C^L(0) \cap \rho$ is not open in ρ^E. Hence $C^L(0)$ is not open in M^A.

Similarly, space like straight line, time like straight line, time like straight line, x-axis, y-axis and singletons are not open in M^A.

Proposition 3.7: *Let M^A be the n-dimensional Minkowski space with A-topology. Then* $C^L(0)$ *is closed in* M^A.

Proof: Let $X \equiv C^L(0)$. Then $X^c = M - C^L(0) = \{C^S(0) \cup C^T(0)\} - \{0\}$. This implies $X^c = (C^S(0)$- $\{0\}) \cup (C^T(0)$- $\{0\})$. Since $(C^S(0)$-$\{0\})$ and $(C^T(0)$- $\{0\})$ are open in M^A by Proposition 3.2 and arbitrary union of open sets is open, hence X^c is open in M^A. This implies that $X \equiv C^L(0)$ is closed in M^A.

The other closed sets of M^A are space like straight line, time like straight line, light like straight line, x-axis, y-axis and singletons.

Proposition 3.8: *Let M^A be the n-dimensional Minkowski space with A-topology. Then* $C^L(0)$ - $\{0\}$, $C^T(0)$ - $\{0\}$ *and* $C^S(0)$ - $\{0\}$ *are not closed in* M^A.

Proof: We have proved only that $C^L(0) - \{0\}$ is not closed in M^A. Let μ be the light-like line passing through origin. Since $(C^L(0) - \{0\}) \cap \mu = \mu$ - $\{0\}$ which is not closed in μ^E. It follows that $(C^L(0) - \{0\}) \cap \mu$ is not closed in μ^E. Hence $C^L(0) - \{0\}$ is not closed in M^A.

Similarly, $C^T(0)$ and $C^S(0)$ are not closed in M^A.

Proposition 3.9: *Let $\{t_n\}$ be a sequence of distinct time-like lines (or space-like hyper-plane) passing through a point z. Let $z_n \in t_n$ such that $d(z_n, z) \to 0$. Then the set $Z = \{z_k: z_k \neq z, k \in N\}$ is closed in M^A* [6].

Proposition 3.10: *Let M be the n-dimensional Minkowski space and $Z = \{z_k: z_k \neq z, k \in N\}$ be a set of points on distinct time-like lines passing through a point z such that $d(z_n, z) \to 0$. Then the set M - Z is not open in M^t.*

Proof: Since every basic open set $N_\varepsilon^t(z)$ in *t*-topology about *z* will meet *z* [5], therefore $N_\varepsilon^t(z) \not\subset M - Z$, $M - Z$ is not open in M^t.

Proposition 3.11: *Let M be the n-dimensional Minkowski space and $Z = \{z_k: z_k \neq z, k \in N\}$ be a set of points on distinct time-like lines (or space-like hyper-plane) passing through a point z such that $d(z_n, z) \to 0$. Then the set M - Z is not open in M^f.*

4. COMPARISON OF *A*-TOPOLOGY WITH OTHER TOPOLOGIES

In this section, comparison of *A*-topology with other topologies on *M* has been carried out, in detail. Nanda [6] proved that *A*-topology is finer than the Euclidean topology. For completeness, we give the result as Proposition 4.1.

Proposition 4.1: *Let M be the n-dimensional Minkowski space. Then the A-topology on M is finer than the Euclidean topology on M.*

Proof: Let *A* and *E* denote *A*-topology and Euclidean topology respectively. The Euclidean topology induces one-dimensional Euclidean topology on every time-like line and light-like line and three-dimensional Euclidean topology on every space-like hyper-plane on *M*. By definition, *A* is the finest such topology hence, $E \subseteq A$.

Proposition 4.2: *Let M be the n-dimensional Minkowski space. Then the A-topology is strictly finer than the Euclidean topology on M.*

Proof: By Proposition 4.1, *A*-topology is finer than the Euclidean topology on *M*. To prove the rest we have to find a subset of *M* which is open in M^A but not open in M^E. For this, let $\{t_n\}$ be a sequence of distinct time-like lines passing through a point *z*. Let $z_n \in t_n$ such that $d(z_n, z) \to 0$. Let $Z = \{z_k: z_k \neq z, k \in N\}$. Then the set *Z* is closed in M^A by Proposition 3.9. This implies $(M - Z)$ is open in M^A. On other hand, $(M - Z)$ is not open in M^E because for any $\varepsilon > 0$, every open ball $N_\varepsilon^E(z)$ intersects *Z*, this implies $N_\varepsilon^E(z) \not\subset (M - Z)$.

Proposition 4.3: *Let M be the n-dimensional Minkowski space. Then the A-topology on M is not comparable with the t-topology.*

Proof: It is known that $C^T(0)$ is open in M^t [1]. By Proposition 3.6, $C^T(0)$ is not open in M^A. This implies that t topology is not coarser then A-topology. Further, let $\{t_n\}$ be a sequence of distinct time-like lines passing through a point z. Let $z_n \in t_n$ such that $d(z_n, z) \to 0$. Let $Z = \{z_k: z_k \neq z, k \in N\}$. Then the set Z is closed in M^A by Proposition 3.9. This implies $(M - Z)$ is open in M^A, whereas $(M - Z)$ is not open in the t-topology Proposition 3.10. This proves that A-topology is not coarser then t-topology.

Proposition 4.4: *Let M be the n-dimensional Minkowski space. Then the A-topology is not comparable with the s-topology on M.*

Proof: Let A and s denote A-topology and s-topology respectively. To prove that the A-topology is not comparable with s-topology, we prove that $A \not\subset s$ and $s \not\subset A$. Since $C^S(0)$ is open in M^s [5] but not open in M^A by Proposition 3.6. This implies that s-topology is not coarser then A-topology. Further, let $\{s_n\}$ be a sequence of distinct space-like hyper-planes passing through a point z. Let $z_n \in s_n$ such that $d(z_n, z) \to 0$. Let $Z = \{z_k: z_k \neq z, k \in N\}$. Then the set Z is closed in M^A by Proposition 3.9. This implies $(M - Z)$ is open in M^A. On other hand, $(M - Z)$ is not open in the s-topology by Proposition 3.11. This proves that A-topology is not coarser then s-topology.

Proposition 4.5: *Let M be the n-dimensional Minkowski space. Then the A-topology is not comparable with the f-topology on M.*

Proof: Let A and f denote A-topology and f-topology respectively. To prove that the A-topology is not comparable with f-topology, we have to show that $A \not\subset f$ and $f \not\subset A$. Since $C^T(0)$ is open in M^f [8] but not open in M^A by Proposition 3.6. This implies that $f \not\subset A$. Further, $Z = \{z_k: z_k \neq z, k \in N\}$ be a set of points on distinct time-like lines passing through a point z such that $d(z_n, z) \to 0$. Then the set Z is closed in M^A by Proposition 3.9. This implies $(M - Z)$ is open in M^A [6]. On other hand, $(M - Z)$ is not open in the f-topology by Proposition 3.11. Hence $A \not\subset f$.

Proposition 4.6: *Let M be the n-dimensional Minkowski space. Then the A-topology is coarser than the space topology on M.*

Proof: Let A and S denote A-topology and space topology respectively. To prove that the A-topology is coarser than the space topology, we have to show that $A \subseteq S$. The A-topology on M induces three-dimensional Euclidean topology on every space-like hyper-plane and space topology is the finest such topology [4], hence $A \subseteq S$.

Proposition 4.7: *Let M be the n-dimensional Minkowski space. Then the A-topology is coarser than the time topology on M.*

Proof: Let A and T denote A-topology and time topology respectively. To prove that the A-topology is coarser than time topology, we have to prove that $A \subseteq T$. The A-topology on M induces one-dimensional Euclidean topology on every time-like line and time topology is the finest topology that induces one-dimensional Euclidean topology on every time-like line [4], hence $A \subseteq T$.

Proposition 4.8: *Let M be the n-dimensional Minkowski space. Then the A-topology is coarser than the fine topology on M.*

Proof: Let A and F denote A-topology and fine topology respectively. To prove that the A-topology is coarser than the Fine topology on M, we prove that $A \subseteq F$. The A-topology induces one-dimensional Euclidean topology on every time-like line and three-dimensional Euclidean topology on every space-like hyper-plane and fine topology is finest such topology [8], hence $A \subseteq F$.

5. CONCLUSION

The A-topology on Minkowski space is a physically significant topology. It induces Euclidean topology on light rays which shows that paths of photons are continuous. In the present work, it has been obtained that A-topology is coarser than time, space and fine topologies, but not comparable with *t, s* and *f*- topologies.

Basically, study and visualization of open sets and closed sets of M^A are a must for further researches in M^A. Open sets are used while comparing A-topology with other topologies. It's comparison with other topologies is important because M^A reflects some topological properties which are preserved under finer or coarser topologies. Because of the complexity of the nature of A-topology, it can be concluded that the present study is important from both mathematical and physical point of view.

REFERENCES

[1] Agrawal, G. & Shrivastava, S., *t-topology on the n-dimensional Minkowski Space*, J. Math. Phys., 50, 053515 (2009).

[2] Dossena, G., *Some results on the Zeeman topology*, J. Math. Phys. 48, 113507 (2007) .

[3] Munkres, J. R., *Topology - A First Course*, Pearson Education Inc., Singapore. 2000.

[4] Nanda, S., *Topology for Minkowski Space*, J. Math. Phys. 12, 394 (1971).

[5] Nanda, S., *Weaker Versions of Zeeman's Conjectures on Topologies for Minkowski Space*, J. Math. Phys. 13, 12 (1972).

[6] Nanda, S., *A-Topology for Minkowski Space*, J. Austral. Math. Soc. 21, 53 (1979).

[7] Zeeman, E. C., *The Topology of Minkowski Space*, Topology 6.