Some Common Fixed Point Theorem under Rational Expressions in Cone Metric Space

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ABSTRACT

The aim of this paper is to prove some common fixed point theorem in cone metric space for rational expression under normal cone setting. Our result generalize the main result of Jaggi & Das, Gupta.

Keywords: Cone metric space common fixed point, metric space, normal cone, rational expression.

AMS Classification: 47H10, 54H25, 55M20

1. INTRODUCTION

Fixed point theory plays a basic role in application of various branches of mathematics from elementary calculus and linear algebra to topology and analysis. Fixed point theory is not restricted to mathematics and this theory has many applications in other discipline. The Banach Contraction principle with rational expressions have been expanded and some common fixed point theorem have been obtained in [1] [2]. Cone metric space where consider by Huang and Zhang [4] who reintroduced the concept which has been known since the middle of 20^{th} century. They have considered convergent in cone metric space, introduced completeness of cone metric space and prove a Banach contraction mapping theorem and some other fixed point theorems involving contractive type mapping in cone metric space using the normality condition. Our results generalized the main result of Jaggi [3], Das & Gupta [11] with adding new mappings.

2. PRELIMINARY:

Let G be a real Banach space and $'K'$ a subset at G. K is called a cone iff

i. K is closed, nonempty, and $K \neq \{0\}$

ii. $a, b \in R, a, b \geq 0, x, y \in K \Rightarrow ax + by \in K.$

$iii.\ \ x \in K\ and\ -x \in K \Longrightarrow x = 0\ i.e.K \cap (-K) = \{0\}.$

Given a cone $K \subset G$, we define a partial ordering $\leq$ with respect to

$K\ by\ x \leq y\ iff\ y - x \in K$
We write $x < y\ if\ x \leq y\ but\ x \neq y.$
While $x \ll y\ if\ y - x \in int\ K.$
The cone K is called normal if there is a number $M > 0\ s.t. x, y \in G.$
$0 \leq x < y\ implies\ \| x \| \leq M \| y \|.$
The least positive number satisfying above is called the normal constant of .

Definition: 2.1 [2]

Let x be non-empty set G is a real Banach space and $K \subset G,$ a cone. Suppose the mapping $d: X \times X \rightarrow G$ satifies

$d\ 1.0 < d\ (x, y) for\ all\ x, y \in X\ and\ d\ (x, y) = 0\ iff\ x = y$

$d\ 2. d\ (x, y) = d\ (y, x) for\ all\ x, y \in X$

$d\ 3. d\ (x, y) \leq d(x, z) + d(z, y) for\ all\ x, y, z \in X$

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Definition 2.2 [4]
Let (X, d) is said to be a complete cone metric space, if every Cauchy sequence is convergent in .

Definition 2.3 [9]
Let (X, d) be a cone metric space a self mapping T on X is called an jaggi contraction, if it satisfies the condition

$$d(Tx, Ty) \leq \alpha\ \frac{d(x,Tx)d(y,Ty)}{d(x,y)} + \beta\ d\ (x, y) + L \min\{d(x, Ty), d(y, Tx)\}$$

$\forall\ x, y \in X, Where\ L \geq 0\ and\ \alpha, \beta \in [0, 1)\ with\ \alpha + \beta < 1.$

Definition 2.4 [11]

[9] Let (X, d) be a cone metric space a self mapping T on X is called an Das & Gupta contraction, if it satisfies the condition.

$d(Tx, Ty) \leq \alpha \frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)} + \beta d(x,y) + L\, min. \{ d(x, Tx,), d(x, Ty), d(y, Tx)\}$

$\forall x, y \in X, Where\ L \geq 0\ and\ \alpha, \beta \in [0,1)\ with\ \alpha + \beta < 1.$

3.0 Main Result:-

Theorem 3.1 Let (X, d) be a complete cone metric space and K be a normal cone with normal constant M. Let $T: X \to X$.
$d(Tx, Ty) \leq$
$\alpha \frac{d(x,Tx)d(y,Ty)}{d(x,y)} + \beta \frac{d(x,Tx)d(y,Ty) + d(y,Ty)d(y,Tx)}{d(x,y)} + \gamma d(x,y) + L \min\{d(x, Ty), d(y, Tx)\}$
For all $x, y \in X\ where\ L \geq 0\ and\ \alpha, \beta, \gamma \in [0,1)\ with\ \alpha + 2\beta + 2\gamma < 1$. Then T has a unique fixed point in X. ……… $3.1.a$
Proof:- Choose $x_0 \in X, Set\ x_1 = Tx_0,\ x_n = Tx_{n-1}$.

$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq$

$$\alpha \frac{d(x_{n-1}, Tx_{n-1})d(x_n, Tx_n)}{d(x_{n-1}, x_n)} + \beta \frac{d(x_{n-1}, Tx_{n-1})d(x_n, Tx_n) + d(x_n, Tx_n)d(x_n, Tx_{n-1})}{d(x_{n-1}, x_n)} + \gamma d(x_{n-1}, x_n) + L \min\{d(x_{n-1}, Tx_n), d(x_n, Tx_{n-1})\}$$

$$\leq \alpha \frac{d(x_{n-1}, x_n)d(x_n, x_{n+1})}{d(x_{n-1}, x_n)} + \beta \frac{d(x_{n-1}, x_n)d(x_n, x_{n+1}) + d(x_n, x_{n+1})d(x_n, x_n)}{d(x_{n-1}, x_n)} + \gamma d(x_{n-1}, x_n) + L \min\{d(x_{n-1}, x_{n+1}), d(x_n, x_n)\}$$

$$= \alpha d(x_n, x_{n+1}) + \beta d(x_n, x_{n+1}) + \gamma d(x_{n-1}, x_n)$$

$$d(x_n, x_{n+1}) \leq \frac{\gamma}{1 - \alpha - \beta} d(x_{n-1}, x_n)$$

$$d(x_n, x_{n+1}) \leq s\, d(x_{n-1}, x_n)$$

Where $s = \frac{\gamma}{1-\alpha-\beta}, \alpha + 2\beta + 2\gamma < 1, 0 < K < 1$ and by induction

$$\leq s^n d(x_{n-1}, x_n)$$

by triangle inequality

$$d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots\ldots + d(x_{n+m-1}, x_m)$$

$$\leq (s^n + s^{n+1} + s^{n+2} + \ldots\ldots + s^{n+m-1})d(x_0, x_1)$$

$\leq \frac{s^n}{1-s} d(x_0, Tx_0)$

Now $\| d(x_n, x_m) \| \leq M \frac{s^n}{1-s} \| d(x_0, x_1) \|$

Which implies that $d(x_n, x_m) \to 0$ as $n \to \infty$. Hence x_n is a Cauchy sequence, so by completeness of X this sequence must be convergent in X. Now w is another point of X

$d(w, Tw) \leq d(w, x_{n+1}) + d(x_{n+1}, Tw)$

$\leq d(w, x_{n+1}) + d(Tx_n, Tw)$

$\leq d(w, x_{n+1}) + \alpha \frac{d(x_n,Tx_n)d(w,Tw)}{d(x_n,w)} + \beta \frac{d(x_n,Tx_n)d(w,Tw) + d(w,Tw)d(w,Tx_n)}{d(x_n,w)} + \gamma d(x_n, w) +$
$L \min\{d(x_n, Tw), d(w, Tx_n)\}$

$$\leq d(w, x_{n+1}) + \gamma d(x_n, w) + L \min\{d(x_n, Tw), d(w, x_{n+1})\}$$

So using the condition of normality of cone.

$$\| d(w, Tw) \| \leq M [\| d(w, x_{n+1}) \| + \gamma \| d(x_n, w) \| + L \min\{\| d(x_n, Tw), d(w, x_{n+1}) \|\}]$$

As $n \to 0$ we have

$\| d(w, Tw) \| \leq 0$, Hence $w = Tw, w$ is a fixed point of T.

Remarks:
1. We put $L = \beta = 0$ in theorem 3.1 we get Jaggi result.

Theorem 3.2
Let (X, d) be a complete cone metric space and K normal cone with normal constant M. Let $T: X \to X$ be a

$d(Tx,Ty) \leq$
$\alpha \frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)} + \beta \frac{d(y,Tx)[1+d(x,Ty)]}{1+d(x,y)} + \gamma\, d\,(x,y) + L\, min.\{\, d(x,Tx,), d\,(x,Ty), d\,(y,Tx)\}$

$\forall\, x,y\ \in X, Where\ L \geq 0\ and\ \alpha,\beta,\gamma\ \in [0,1)\ with\ \alpha+\beta+\gamma\ < 1$. Then T has a unique fixed point in X.

Proof:- Choose $x_0 \in X$, set $x_1 = Tx_0, x_n\ = Tx_{n-1}$

$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n)$

$\leq$
$\alpha \frac{d(x_n,Tx_n)[1+d(x_{n-1},Tx_{n-1})]}{1+d(\,x_{n-1},x_n)} + \beta \frac{d(x_n,Tx_{n-1})[1+d(x_{n-1},Tx_n)]}{1+d(\,x_{n-1},x_n)} + \gamma\, d\,(x_{n-1},x_n) +$
$L\, min.\{\, d(x_{n-1},Tx_{n-1},), d\,(x_{n-1},Tx_n), d\,(x_n,Tx_{n-1})\}$

$\leq$
$\alpha \frac{d(x_n,x_{n+1})[1+d(x_{n-1},x_n)]}{1+d(\,x_{n-1},x_n)} + \beta \frac{d(x_n,x_n)[1+d(x_{n-1},x_{n+1})]}{1+d(\,x_{n-1},x_n)} + \gamma\, d\,(x_{n-1},x_n) +$
$L\, min.\{\, d(x_{n-1},x_n), d\,(x_{n-1},x_{n+1}), d\,(x_n,x_n)\}$

$\leq \alpha \frac{d(x_n,x_{n+1})[1+d(x_{n-1},x_n)]}{1+d(\,x_{n-1},x_n)} + \gamma\, d\,(x_{n-1},x_n)$

$\leq \alpha\, d(x_n,x_{n+1}) + \gamma\, d\,(x_{n-1},x_n)$

$d(x_n,x_{n+1}) \leq \frac{\gamma}{1-\alpha}\, d\,(x_{n-1},x_n)$

$d(x_n,x_{n+1}) \leq\ s\, d\,(x_{n-1},x_n)$

Where $s = \frac{\gamma}{1-\alpha} < 1., Where\ \alpha + \beta + \gamma\ < 1\ \&\ 0\ < s < 1$

$d(x_n,x_{n+1}) \leq\ s^n\, d\,(x_0,x_1)$

by triangle inequality

$d(x_n,x_m) \leq d(x_n,x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ \ldots\ldots + d\,(x_{n+m-1},x_m)$

$$\leq (s^n + s^{n+1} + s^{n+2} + \ \ldots\ldots + s^{n+m-1})d(x_0,x_1)$$

$\leq \frac{s^n}{1-s} d(x_0, Tx_0)$

So that Now $\| d(x_n, x_m) \| \leq M \frac{s^n}{1-s} \| d(x_0, x_1) \|$

Which implies that $d(x_n, x_m) \to 0$ as $n \to \infty$. Hence x_n is a Cauchy sequence so, by completeness of X, sequence must be convergent in X.

Now Suppose any other point $w \ in \ X$.

$d(w, Tw) \leq d(w, x_{n+1}) + d(x_{n+1,} Tw)$

$\leq d(w, x_{n+1}) + d(Tx_{n,} Tw)$

$\leq$
$d(w, x_{n+1}) + \alpha \frac{d(w,Tw)[1+d(x_{n,},Tx_{n,})]}{1+d(x_{n,},w)} + \beta \frac{d(w,Tx_{n,})[1+d(x_{n,},Tw)]}{1+d(x_{n,},w)} + \gamma\, d(x_{n,}, w) +$
$L\, min. \{ d(x_{n,}, Tx_{n,},), d(x_{n,}, Tw), d(w, Tx_{n,})\}$

$\leq d(w, x_{n+1}) + \beta(w, x_{n+1}) + \gamma\, d(x_{n,}, w) + L\, min. \{d(x_{n,}, x_{n+1,}), d(x_{n,}, w), d(w, x_{n+1,})\}$

$\leq (1+\beta) d(w, x_{n+1}) + \gamma\, d(x_{n,}, w) + L\, min. \{d(x_{n,}, x_{n+1,},), d(x_{n,}, w), d(w, x_{n+1,})\}$

So condition of normality of cone.

$\| d(w, Tw) \| \leq M[(1+\beta) \| d(w, x_{n+1}) \| + \gamma \| d(x_{n,}, w) \|$
$+ L\, min. \{\| d(x_{n,}, x_{n+1,},), d(x_{n,}, w), d(w, x_{n+1,}) \|\}]$

as $n \to 0$ we have $\| d(w, Tw) \| \leq 0$

Hence $w = Tw, w$ is a fixed point of T.

Remarks:-
If we put $\beta = L = 0$ in theorem 3.2 then we get Das & Gupta rational contraction result.

Theorem 3.3
Let (X, d) be a complete metric space and K a normal cone with normal constant M. Let $S, T: X \to X$ be a

$d(Sx, Ty) \leq$
$\alpha \frac{d(y,Ty)[1+d(x,Sx)]}{1+d(x,y)} + \beta \frac{d(y,Sx)[1+d(x,Ty)]}{1+d(x,y)} + \gamma\, d\,(x, y) + \ L\, min.\{\, d(x, Sx,), d\,(x, Ty), d\,(y, Sx)\}$

For all $x, y \in X$ where $L \geq 0$ and $\alpha, \beta, \gamma \in [0,1)$ with $\alpha + \beta + \gamma < 1$. Then S & T be a unique fixed point in X.

Proof: Choose $x_1 \in Sx_0$ & $x_2 = T\, x_1$

s.t. $x_{2n+1} = Sx_{2n}$ and $x_{2n+2} = Tx_{2n+1}$

Now

$d(x_{2n+1}, x_{2n+2}) = d\,(Sx_{2n}, Tx_{2n+1})$

$\leq \alpha \frac{d(x_{2n+1},Tx_{2n+1})[1+d(x_{2n},Sx_{2n})]}{1+d(x_{2n},x_{2n+1})} + \beta \frac{d(x_{2n+1},Sx_{2n})[1+d(x_{2n},Tx_{2n+1})]}{1+d(x_{2n},x_{2n+1})} + \gamma\, d\,(x_{2n}, x_{2n+1}) +$
$L\, min.\{\, d(x_{2n}, Sx_{2n},), d\,(x_{2n}, Tx_{2n+1}), d\,(x_{2n+1}, Sx_{2n})\}$

$\leq \alpha \frac{d(x_{2n+1},x_{2n+2})[1+d(x_{2n},x_{2n+1})]}{1+d(x_{2n},x_{2n+1})} + \beta \frac{d(x_{2n+1},x_{2n+1})[1+d(x_{2n},x_{2n+2})]}{1+d(x_{2n},x_{2n+1})} + \gamma\, d\,(x_{2n}, x_{2n+1}) +$
$L\, min.\{\, d(x_{2n}, x_{2n+1},), d\,(x_{2n}, x_{2n+2}), d\,(x_{2n+1}, x_{2n+1})\}$

$\leq \alpha d(x_{2n+1}, x_{2n+2}) + \gamma\, d\,(x_{2n}, x_{2n+1})$

$d(x_{2n+1}, x_{2n+2}) \leq \alpha d(x_{2n+1}, x_{2n+2}) + \gamma\, d\,(x_{2n}, x_{2n+1})$

$$d(x_{2n+1}, x_{2n+2}) \ \leq \frac{\gamma}{1-\alpha} d\,(x_{2n}, x_{2n+1})$$

$\leq S\, d\,(x_{2n}, x_{2n+1})$

Where $S = \frac{\gamma}{1-\alpha} < 1$ because $\alpha + \beta + \gamma < 1$

Similarly

$d(x_{2n+2}, x_{2n+3}) = d\,(Sx_{2n+1}, Tx_{2n+2})$

$$\leq \alpha \frac{d(x_{2n+2}, Tx_{2n+2})[1 + d(x_{2n+1}, Sx_{2n+1})]}{1 + d(x_{2n+1}, x_{2n+2})} + \beta \frac{d(x_{2n+2}, Sx_{2n+1})[1 + d(x_{2n+1}, Tx_{2n+2})]}{1 + d(x_{2n+1}, x_{2n+2})} + \gamma\, d(x_{2n+1}, x_{2n+2}) + L\, min. \{ d(x_{2n+1}, Sx_{2n+1},), d(x_{2n+1}, Tx_{2n+2}), d(x_{2n+2}, Sx_{2n+1})\}$$

$$\leq \alpha \frac{d(x_{2n+2}, x_{2n+3})[1 + d(x_{2n+1}, x_{2n+2})]}{1 + d(x_{2n+1}, x_{2n+2})} + \beta \frac{d(x_{2n+2}, x_{2n+2})[1 + d(x_{2n+1}, x_{2n+3})]}{1 + d(x_{2n+1}, x_{2n+2})} + \gamma\, d(x_{2n+1}, x_{2n+2}) + L\, min. \{ d(x_{2n+1}, x_{2n+2},), d(x_{2n+1}, x_{2n+3}), d(x_{2n+2}, x_{2n+2})\}$$

$$\leq \alpha\, d(x_{2n+2}, x_{2n+3}) + \gamma\, d(x_{2n+1}, x_{2n+2})$$

$$d(x_{2n+2}, x_{2n+3}) \leq \alpha\, d(x_{2n+2}, x_{2n+3}) + \gamma\, d(x_{2n+1}, x_{2n+2})$$

$$d(x_{2n+2}, x_{2n+3}) \leq \frac{\gamma}{1-\alpha} d(x_{2n+1}, x_{2n+2})$$

$$d(x_{2n+2}, x_{2n+3}) \leq s\, d(x_{2n+1}, x_{2n+2})$$

where $S = \frac{\gamma}{1-\alpha} < 1, because\ \alpha + \beta + \gamma < 1$ by triangle inequality

$$d(x_{2n+2}, x_{2n+3}) \leq d(x_{2n+2}, x_{2n+1}) + d(x_{2n+1}, x_{2n+3})$$

so that

$$d(x_{2n+2}, x_{2n+3}) \leq s\, d(x_{2n+1}, x_{2n+2})$$

$$\therefore d(x_{n+1}, x_{n+2} \leq s\, d(x_n, x_{n+1}) \leq \ldots\ldots\ldots\ldots\ldots \leq s^n (x_0, x_1)$$

by triangle inequality

$$d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots\ldots\ldots\ldots + d(x_{m-1}, x_m)$$

$$\leq (s^n + s^{n+1} + \ldots\ldots\ldots + s^{m-1})\{d(x_0, x_1)\}$$

$$\leq \frac{s^n}{1-s} d(x_0, x_1)$$

we get

$$\| d(x_n, x_{n+1}) \| \leq M \frac{s^n}{1-s} \| d(x_0, x_1) \|$$

which implies that $d(x_n, x_{n+1}) \rightarrow 0 \; as \; n \rightarrow \infty$.

Hence $\{x_n\}$ is a Cauchy sequence, so by completeness of X. Sequence must be convergent in X.

Now we prove that w is a common fixed point of $S \; and \; T$.

$d(w, Tw) \leq d(w, x_{2n+1}) + d(x_{2n+1}, Tw)$

$\leq d(w, x_{2n+1}) + d(Sx_{2n}, Tw)$

$\leq d(w, x_{2n+1}) + \alpha \frac{d(w,Tw)[1+d(x_{2n},Sx_{2n})]}{1+d(x_{2n},w)} + \beta \frac{d(w,Sx_{2n})[1+d(x_{2n},Tw)]}{1+d(x_{2n},w)} + \gamma \, d\,(x_{2n}, w) +$
$L \, min.\{ d(x_{2n}, Sx_{2n},), d\,(x_{2n}, Tw), d\,(w, Sx_{2n})\}$

$\leq d(w, x_{2n+1}) + \alpha \frac{d(w,w)[1+d(x_{2n},x_{2n+1})]}{1+d(x_{2n},w)} + \beta \frac{d(w,x_{2n+1})[1+d(x_{2n},w)]}{1+d(x_{2n},w)} + \gamma \, d\,(x_{2n}, w) +$
$L \, min.\{ d(x_{2n}, x_{2n+1},), d\,(x_{2n}, w), d\,(w, x_{2n+1})\}$

$\leq d(w, x_{2n+1}) + \beta d(w, x_{2n+1}) + \gamma \, d\,(x_{2n}, w)$
$+ L \, min\{ d(x_{2n}, x_{2n+1},), d\,(x_{2n}, w), d(w, x_{2n+1})\}$

so using the condition normality of cone.

$\| d(w, Tw) \| \leq M[\| d(w, x_{2n+1}) \| + \beta \| d(w, x_{2n+1}) \| + \gamma \; \| d\,(x_{2n}, w) \| \; + L \, min.\{ \|$
$d(x_{2n}, x_{2n+1},), d\,(x_{2n}, w), d\,(w, x_{2n+1}) \| \}]$

as $n \rightarrow 0$

we have

$\| d(w, Tw) \| \leq 0$

so we get $w = Tw$, Similarly $w = Sw$

therefore w is a fixed point of S & T.

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