The Graphics of Some Non Quadratic Rational Maps

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ABSTRACT

Many physical and biological systems are well represented by some dynamical models. The Julia sets are of prime importance in exploring the characteristics of such models. Our aim is to study the Julia sets for a non quadratic rational map using Mann iterative procedure.

Keywords: Julia sets; non quadratic rational map; Mann iteration.

1. INTRODUCTION

In late seventies of twentieth century, Benoit Mandelbrot first published his famous picture, known as Mandelbrot set and popularized the great work of Julia by adding computer graphics to it in the form of Julia sets. These sets show the patterns based on chaotic behavior of the complex functions and exhibit their resemblance with some natural patterns due to the property of self similarity (see [4] and [5]).

Computer generated eye catching images of Julia sets for different complex functions attract the researchers and the subsequent results can be seen in the form of a series of papers. In almost all such papers, authors study Julia sets for different functions like exponential, logarithmic, transcendental etc and explored the different interesting properties regarding convergence pattern of the generated Julia sets (for example see [3] and [8]- [10]). Ereneko [3] studied Julia sets for transcendental functions and observed that there is at least one escape point which always lies in the Julia sets. Different fractal patterns of Julia and Mandelbrot sets are widely studied by Aggarwal et al [1], Prasad and Katiyar [9-12], Prasad and Mishra [13] and Rani and Kumar [14] for Picard, Mann, Ishikawa and other iterative schemes. They studied the complex dynamics of different functions by selecting an appropriate escape criterion for such functions. The intent of this chapter is to study the fractal graphics of the superior Julia sets of the non quadratic rational map and establish the convergence criteria.

2. PRELIMINARIES

Definition 2.1 [5]. Let f: $C \rightarrow C$ be a polynomial of degree greater than 1, where C is set of complex numbers. Let F_f be the set of points in C, whose orbit does not converge to the points at

∞, i.e. $F_f = \{z \in C$: { $|f_n(z)| < M$ } for $n = 0$ to ∞, is bounded$\}$, where f_n is the n^{th} iteration of the complex function f. Then the set F_f is called filled in Julia set associated with the function f and the boundary of F_f is called the Julia set of the function f and is denoted by J_f.

Definition 2.2 [10]. Let X be a non empty set and f: $X \rightarrow X$. For a Point x_0 in X the Picard orbit is the set of all iterates of a point x_0, that is $O(f, x_0) := \{x_n: x_n = f\, x_{n-1}, n = 1, 2, \ldots\}$.

Definition 2.3 [10]. Let X be a non empty set and f: $X \rightarrow X$. For a Point x_0 in X, construct a sequence $\{x_n\}$ in the following manner: $x_n = r_n f(x_{n-1}) + (1- r_n)\, x_{n-1}$, for $n = 1, 2, 3, \ldots$, where, $0 \leq r_n < 1$ and $\{r_n\}$ is convergent away from 0. Then sequence $\{x_n\}$ is called the Mann orbit of all iterates of a point x_0. In our study, we take $r_n = r$ for all n.

2.1. Julia sets for rational functions

The Julia sets have been extensively studied in the literature for different functions such as trigonometric, logarithmic and exponential functions in different domains. For a detailed discussion on the history and development of these sets one may refer Peitgen et al [6], Devaney[2] and Poon [7] and several references thereof . It is observed that any complex analytic function has a Julia set. As with the quadratic family, the Julia sets for other complex functions are often fractals. Mandelbrot [5] studied Julia sets for some non quadratic rational maps and found some interesting patterns.

2.2. Lattes map

Lattes observed that by changing the variable $z = \rho(\theta)$, where $\rho(\theta)$ is a special case of Weierstrass elliptic function and using map $\theta \rightarrow 2\theta$, map $W(z) = \frac{1}{4}\frac{(1+z^2)^2}{z(z^2-1)}$, can be easily obtained**,** which is chaotic on its Julia set (for detail see [5]).

3. MAIN RESULTS

3.1. Convergence of the map $W(z) = \frac{\lambda}{4}\frac{(1+z^2)^2}{z(z^2-1)}$: Here, we study convergence of the map $W(z) = \frac{\lambda}{4}\frac{(1+z^2)^2}{z(z^2-1)}$ towards its fixed points u such that $W(u) = u$, for Picard and Mann both iterative schemes with different values of parameter λ and Mann parameters $r_n = r$ for all n. in all our figures form Fig. 1 to Fig. 11, we represent number of iterations along horizontal axis and the value of $W(z)$ at corresponding iterations along vertical axis. We use same initial choice of z_0 for all the calculations.

Table 1. λ **= 0.2+0.3*i*, *r* = 0.8**

Number of Iterations	$\lvert u \rvert$	**Number of Iterations**	$\lvert u \rvert$
61	0.2774	71	0.2774
62	0.2775	72	0.2774
63	0.2772	73	0.2773
64	0.2776	74	0.2774
65	0.2772	75	0.2773
66	0.2776	76	0.2774
67	0.2772	77	0.2773
68	0.2775	78	0.2774
69	0.2774	79	0.2774
70	0.2775	80	0.2774

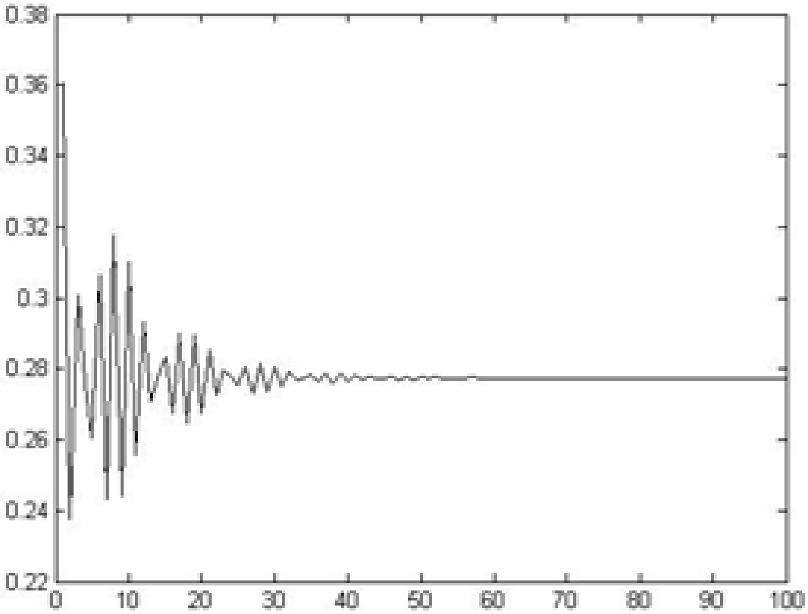


Fig. 1. λ **=0.2+0.3*i*, *r* = 0.8**

Clearly, the function converges to its fixed point after 77 iterations.

Table 2. λ=0.2+0.3*i*, *r* = 0.6

Number of Iterations	$\lvert u \rvert$	Number of Iterations	$\lvert u \rvert$
1	0.3606	11	0.2774
2	0.0845	12	0.2774
3	0.6670	13	0.2774
4	0.2874	14	0.2774
5	0.2687	15	0.2774
6	0.2821	16	0.2774
7	0.2755	17	0.2774
8	0.2779	18	0.2774
9	0.2773	19	0.2774
10	0.2773	20	0.2774

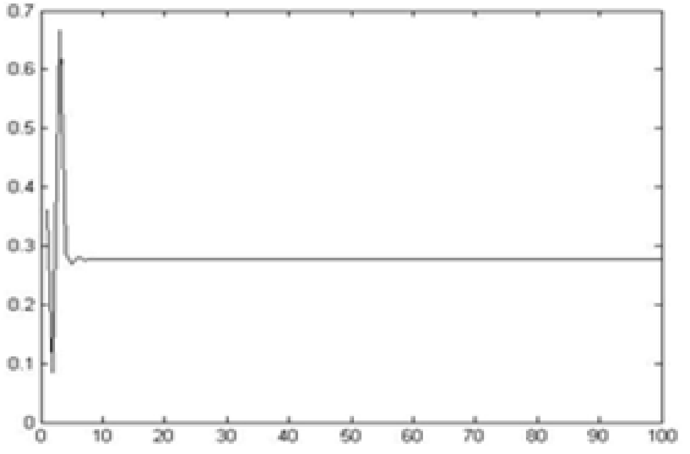


Fig. 2. λ=0.2+0.3*i*, *r* = 0.6

As we can see in Table 2, for Mann iteration with $r = 0.6$ the function $W(z)$ takes only 11 iterations to converge.

Table 3. λ **=0.2+0.3*i*, *r* = 1**

Number of Iterations	$\lvert u\rvert$	Number of Iterations	$\lvert u\rvert$
371	0.3146	381	0.3146
372	0.3146	382	0.3146
373	0.3146	383	0.3145
374	0.3146	384	0.3146
375	0.3145	385	0.3146
376	0.3146	386	0.3145
377	0.3146	387	0.3146
378	0.3146	388	0.3145
379	0.3146	389	0.3146
380	0.3145	390	0.3146

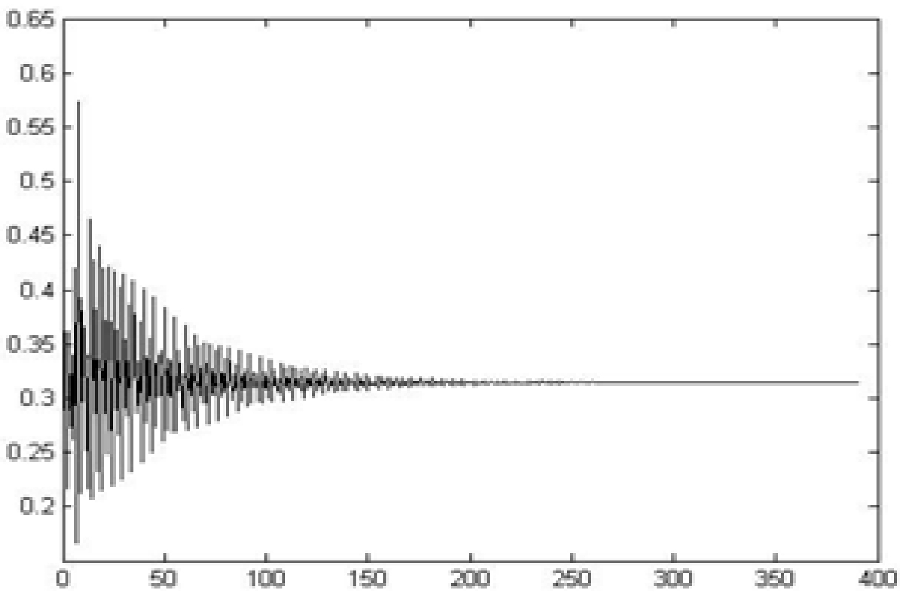


Fig. 3. λ **=0.2+0.3*i*, *r* = 1**

From Table 3, it is clear that the map $W(z)$, for Picard iteration converges after 388 iterations.

Table 4. λ **=0.4+0.8*i*, *r* = 0.4**

Number of Iterations	$\|u\|$	Number of Iterations	$\|u\|$
1	0.8010	11	0.2774
2	0.4839	12	0.2774
3	0.3164	13	0.2774
4	0.2631	14	0.2774
5	0.2726	15	0.2774
6	0.2773	16	0.2774
7	0.2775	17	0.2774
8	0.2774	18	0.2774
9	0.2774	19	0.2774
10	0.2774	20	0.2774

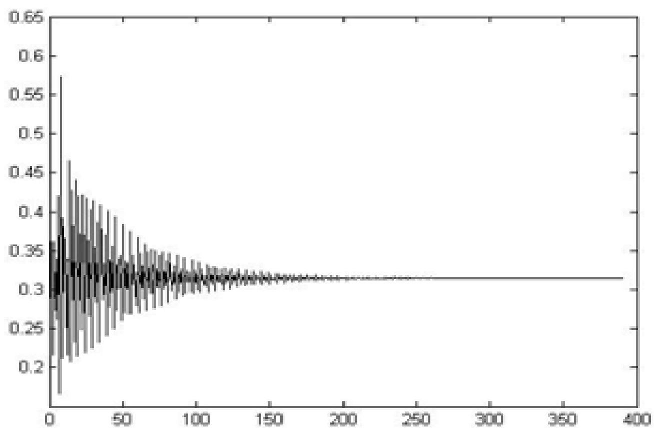


Fig. 4. λ **=0.4+0.8*i*, *r* = 0.4**

Here map converges after 7 iterations.

Table 5. λ **=0.4+0.8*i*, *r* = 1**

Number of Iterations	$\|u\|$	**Number of Iterations**	$\|u\|$
1	0.3146	391	0.3146
2	0.3146	392	0.3146
3	0.3145	393	0.3145
4	0.3146	394	0.3146

5	0.3146	395	0.3146
6	0.3146	396	0.3146
7	0.3146	397	0.3146
8	0.3146	398	0.3146
9	0.3146	399	0.3146
10	0.3146	400	0.3146

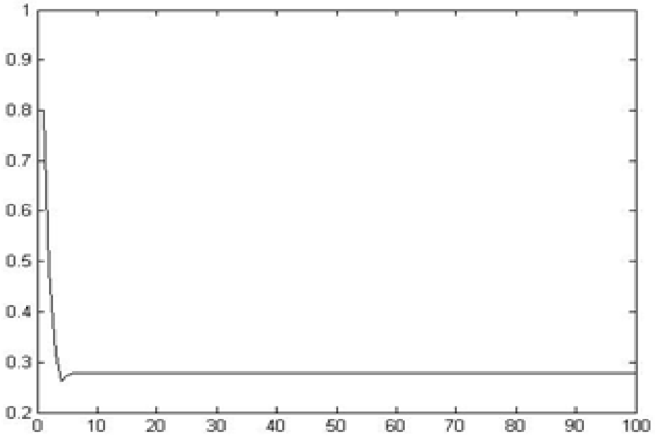


Fig. 5. λ =0.4+0.8i, *r* = 1

Here, the map $W(z)$ converges after 393 iterations with the same initial value as taken in Table 4.

Table 6. λ =-0.2-0.5i, *r* = 0.5

Number of Iterations	$\lvert u \rvert$	Number of Iterations	$\lvert u \rvert$
1	0.5004	11	0.2774
2	0.2700	12	0.2774
3	0.2508	13	0.2774
4	0.2858	14	0.2774
5	0.2763	15	0.2774
6	0.2773	16	0.2774
7	0.2775	17	0.2774
8	0.2774	18	0.2774
9	0.2774	19	0.2774
10	0.2774	20	0.2774

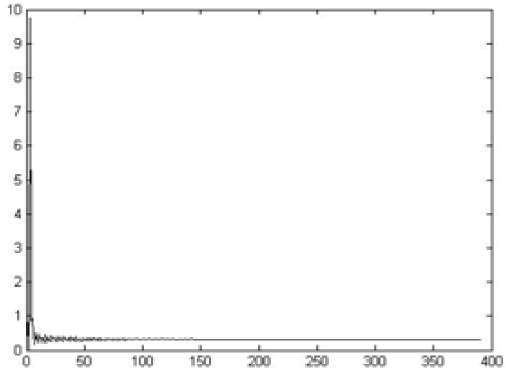


Fig. 6. λ**=-0.2-0.5i,** r **= 0.5**

For λ=-0.2-0.5i, r = 0.5 the map converges after 7 iterations (see Table 6).

Table 7. λ=-0.2-0.5i, r = 0.8

Number of Iterations	$\|u\|$	Number	
of Iterations	$\|u\|$		
1	0.5004	⋮	⋮
2	0.1406	71	0.2773
3	0.5196	72	0.2775
4	0.1850	73	0.2773
5	0.3821	74	0.2774
6	0.2383	75	0.2774
7	0.2886	76	0.2774
8	0.2925	77	0.2774
9	0.2374	78	0.2774
10	0.3346	79	0.2774

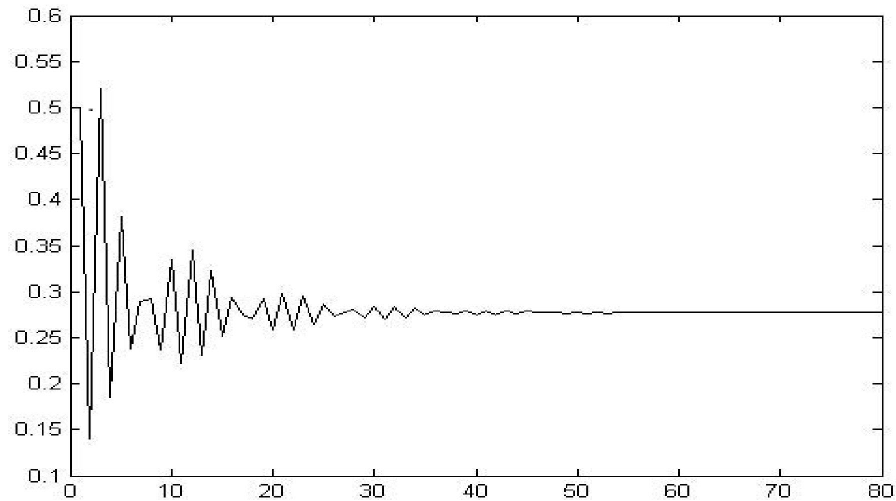


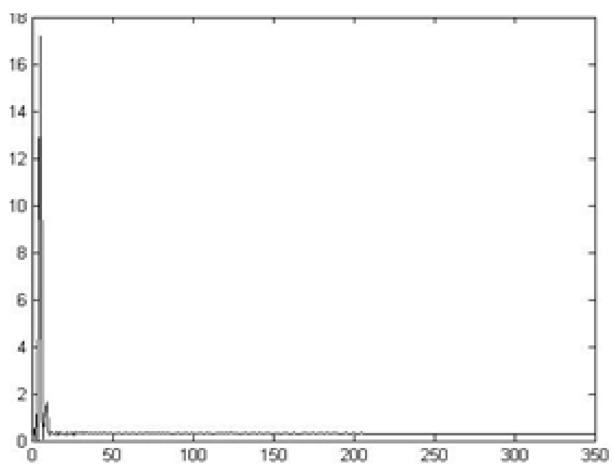
Fig. 7. λ**=-0.2- 0.5*i*, *r*=0.8**

For the same initial value as in Table 6, map converges after 73 iterations at $r = 0.8$.

Table 8. $\lambda = -0.2-0.5i$, $r = 1$

Number of Iterations	$\lvert u \rvert$	Number of Iterations	$\lvert u \rvert$
1	0.5004	⋮	⋮
2	0.0812	342	0.3146
3	1.1225	343	0.3145
4	0.0052	344	0.3146
5	17.1787	345	0.3146
6	1.5641	346	0.3146
7	0.0830	347	0.3146
8	1.0794	348	0.3145
9	1.6583	349	0.3146
10	0.3956	350	0.3146

Fig. 8. $\lambda = -0.2-0.5i$, $r = 1$



For the same initial values as in Table 6 and Table 7, number of iterations needed for convergence are 349 for Picard iterations.

3.2. Generation of some Julia sets for Lattes map using Picard and Mann scheme:

Some patterns of the Julia sets of the Lattes map by varying the values of parameters λ and r are obtained. It is found that for some imaginary values of λ Mann orbit of map produces leaf like pattern as shown in figures. We observe that for some parameters map shows leaf like pattern within 12 iterations.

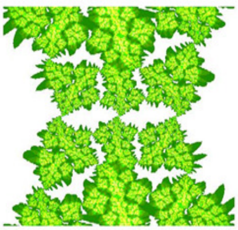


Fig. 12. $r = 0.85$, $\lambda = 2+0.4i$

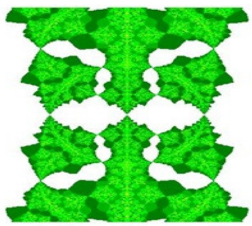


Fig. 13. $r = 1$, $\lambda = 4.4i$

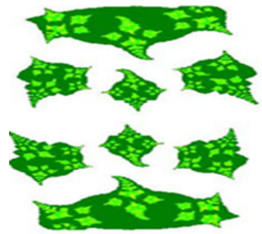


Fig. 14. $r = 1$, $\lambda = 4.4i$

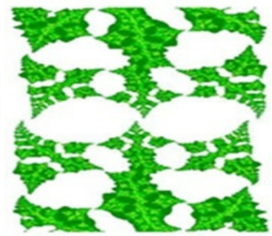


Fig. 15. $r = 0.85$, $\lambda = 3+0.4i$


Fig. 16. $r=0.6$, $\lambda = 4+0.7i$


Fig. 17. $r=0.85$, $\lambda = 3+0.4i$

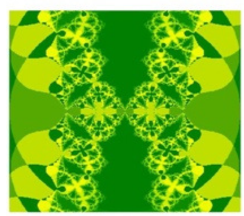


Fig. 18. $r = .5$, $\lambda = 2.4$

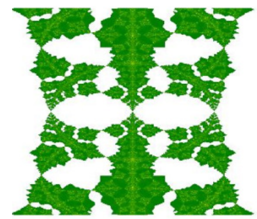


Fig. 19. $r = 0.85$, $\lambda = 2.5$

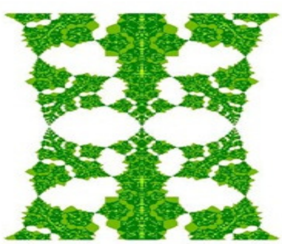


Fig. 20. $r = 1$, $\lambda = 2.5$

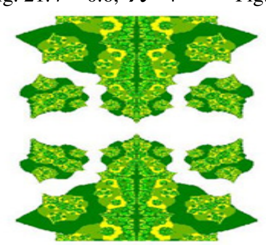


Fig. 21. $r = 0.6$, $\lambda = 4$

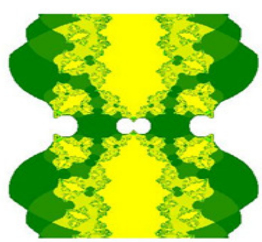


Fig. 22. $r = 0.4$, $\lambda = 4$

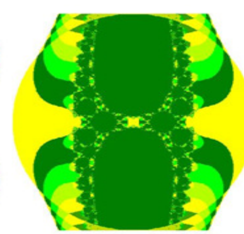


Fig. 23. $r = 0.2$, $\lambda = 4$

4. CONCLUSION

The convergence of Lattes map is studied for Mann and Picard iterative schemes. It is observed that the Mann orbit converges faster than Picard orbit for specific choices of control parameters r (near 0.5). Further, we obtain fractal patterns of Julia sets for these iterative schemes. Some leaves like pattern are obtained for some values of the parameters within very few numbers of iterations. These beautiful patterns are found to be symmetrical about both the co-ordinate axes.

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