# Coupled Fixed Point Results in Complex Valued Metric Spaces

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## ABSTRACT

***In this paper, we have extended and improved the conditions of contraction of previous coupled fixed point theorems in complex valued metric spaces from the constant of contraction to some control functions and established the common coupled fixed point theorems. At the end, we will provide an example in support of our result.***

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## 1. INTRODUCTION AND PRELIMINARIES

The study of metric spaces expressed the most important role to many fields both in pure and applied science such as biology, medicine, physics, and computer science. Many authors generalized and extended the notion of a metric spaces such as vector-valued metric spaces of Perov, G-metric spaces of Mustafa and Sims, cone metric spaces of Huang and Zhang, modular metric spaces of Chistyakov and etc.

In 2011, Azam et al. [1] introduced the notion of complex valued metric space which is a generalization of the classical metric space. They established some fixed point results for mappings satisfying a rational inequality.After that several authors studied many common fixed point results on complex-valued metric spaces (see [2, 3, 4, and 5]). Though complex-valued metric spaces form a special class of cone metric spaces, yet this idea is intended to define rational expressions which are not meaningful in cone metric spaces, and thus many results of analysis cannot be generalized to cone metric spaces. In 2006, Bhaskar et al. [6] introduced the notion of coupled fixed point and proved some fixed point results in this context. Recently, Marwan Amin Kutbi et al. [7] and Shin min kang et al. [8] proved common coupled fixed point theorems for generalized contraction in complex valued metric space.

The aim of this paper is to establish some common coupled fixed-point theorems for nonlinear general contraction mapping in complex-valued metric spaces. Our results generalize recent results of coupled fixed point and result of Kumam et al. about complex valued metric space.

**In order to obtain our results we need to consider the followings.**
Let C be the set of complex numbers and $z_1, z_2 \in$ C. Define a partial order $\precsim$ on C as follows:
$z_1 \precsim z_2$ if and only if $Re(z_1) \leq Re(z_2)$ and $Im(z_1) \leq Im(z_2)$.
It follows that $z_1 \precsim z_2$ if one of the following conditions is satisfied:
$(C1)$ $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$;
$(C2)$ $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$;
$(C3)$ $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$;
$(C4)$ $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.
In particular, we will write $z_1 \precsim z_2$ if $z_1 = z_2$ and one of $(C2)$, $(C3)$ and $(C4)$ is satisfied and we will write $z_1 \precnsim z_2$ if only $(C4)$ is satisfied.

**Remark 1.1.** We obtained that the following statements hold:
1. If $a, b \in$ R with $a \leq b$, then $az \precsim bz$ for all $z \in$ C.
2. If $0 \precsim z_1 \precnsim z_2$, then $|z_1| < |z_2|$.
3. If $z_1 \precnsim z_2$ and $z_2 < z_3$, then $z_1 < z_3$.

**Definition 1.2.** Let $X$ be a nonempty set. Suppose that the mapping $d: X \times X \rightarrow$ C satisfies
(1) $0 \precsim d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, y) \precsim d(x, z) + d(z, y)$ for all $x, y, z \in X$.
Then $d$ is called a *complex valued metric* on $X$ and $(X, d)$ is called a *complex valued metric space.*

**Example 1.3.** Let $X =$ C. Define the mapping $d: X \times X \rightarrow$ C by $d(z_1, z_2) = 2i|z_1 - z_2|$ for all $z_1, z_2 \in X$. Then $(X, d)$ is a complex valued metric space.

**Definition 1.4.** Let $(X, d)$ be a complex valued metric space and $\{x_n\}$ be a sequence in $X$.
(1) If for every $c \in$ C with $0 < c$, there exists $N \in$ N such that $d(x_n, x) < c$ for all $n \geq N$, then $\{x_n\}$ is said to be *convergent* to $x \in X$, and we denote this by $x_n \rightarrow x$ as $n \rightarrow \infty$ or $\lim_{n\rightarrow\infty} x_n = x$.
(2) If for every $c \in$ C with $0 < c$, there exists $N \in$ N such that $d(x_n, x_{n+m}) < c$ for all $n \geq N$, where $m \in$ N, then $\{x_n\}$ is said to be *Cauchy sequence.*
(3) If every Cauchy sequence in $X$ is convergent, then $(X, d)$ is said to be a *complete complex valued metric space*.

**Lemma 1.5.** *Let $(X, d)$ be a complex valued metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \rightarrow 0$ as $n \rightarrow \infty$.*

**Lemma 1.6.** *Let* $(X, d)$ *be a complex valued metric space and* $\{x_n\}$ *be a sequence in X. Then* $\{x_n\}$ *is a Cauchy sequence if and only if* $|d(x_n, x_{n+m})| \to 0$ *as* $n\to\infty$*, where* $m \in$ N.
Now, we introduce the notion of coupled fixed point in complex valued metric space as follows.

**Definition 1.7.** Let $(X, d)$ be a complex valued metric space. Then an element $(x, y) \in X \times X$ is said to be a *coupled fixed point* of the mapping $F: X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$.

**Example 1.8.** Let $X =$ C and define $d : X \times X \to$ C by $d(x, y) = i|x-y|$. Then $(X, d)$ is a complex valued metric space. Consider the mapping $F$ : $X \times X \to X$ with $F(x, y) = i\frac{x+y}{4}$.Here (0, 0) is the coupled fixed point of $F$.

## 2. MAIN RESULTS

Now, we will prove our main result.
**Theorem 2.1.** Let (X, d) be a complete complex valued metric space and $F$: X × X → X be a mapping.If there exist mappings $\Lambda, \Xi$: X × X → [0, 1) such that for all x, y ∈ X:

(i): $\Lambda(F(x,y)) \leq \Lambda(x, y)$ and $\Xi(F(x,y)) \leq \Xi(x, y)$;
(ii): $(\Lambda + \Xi)(x,y) < 1$;
(iii): $d(F(x,y),F(u,v)) \precsim \Lambda(x,y)d(F(x,y),x)+ \Xi(u,v)d(F(u,v), u)$. (2.1)
Then F has a unique coupled fixed point.

*Proof.* Choose $x_0, y_0 \in X$ and set
$x_1 = F(x_0, y_0), y_1 = F(y_0, x_0)$,
• • • ,
$x_{n+1} = F(x_n, y_n), y_{n+1} = F(y_n, x_n)$.
Now,

$$\begin{aligned}
d(x_n, x_{n+1}) &= d(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \\
&\precsim \Lambda(x_{n-1})\, d(F(x_{n-1}, y_{n-1}), x_{n-1}) + \Xi(x_n)\, d(F(x_n, y_n), x_n). \\
&\precsim \Lambda(x_{n-1})\, d(x_n, x_{n-1}) + \Xi(x_n)\, d(x_{n+1}, x_n). \\
&\precsim \Lambda(F(x_{n-2}, y_{n-2}))\, d(x_n, x_{n-1}) + \Xi(F(x_{n-1}, y_{n-1}))\, d(x_{n+1}, x_n). \\
&\precsim \Lambda(x_{n-2})\, d(x_n, x_{n-1}) + \Xi(x_{n-1})\, d(x_{n+1}, x_n). \\
&\ \ \cdots \\
&\ \ \cdots \\
&\precsim \Lambda(x_0)\, d(x_n, x_{n-1}) + \Xi(x_1)\, d(x_{n+1}, x_n). \\
&\precsim \frac{\Lambda(x_0)}{1-\Xi(x_1)} d(x_{n-1}, x_n). \\
&\precsim p\, d(x_{n-1}, x_n).
\end{aligned}$$

Where $p = \frac{\Lambda(x_0)}{1-\Xi(x_1)}$ and $p<1$.
On the same lines we can get,

$d(x_n, x_{n+1}) \precsim pd(x_{n-1}, x_n) \leq p.pd(x_{n-2}, x_{n-1}) \leq .......\leq p^n d(x_o, x_1)$.
Now, for any positive integer m and n with m>n, we have
$d(x_n, x_m) \precsim d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) + ........ + d(x_{m-1}, x_m)$.

$$\precsim p^n d(x_0, x_1) + p^{n+1} d(x_0, x_1) + p^{n+2} d(x_0, x_1) )+........+ p^{m-1} d(x_0, x_1) .$$
$$\precsim (p^n + p^{n+1} + p^{n+2}+........+ p^{m-1})\, d(x_0, x_1) .$$
$$\precsim \frac{p^n}{1-p} d(x_0, x_1) .$$

Therefore,
$$\left|d(x_{n,} x_m)\right| \leq \frac{p^n}{1-p} \left|d(x_{0,} x_1)\right|$$

Since, $p \in [0,1)$, if we taking limit as m, n $\rightarrow 0$, then $|d(x_n, x_m)| \rightarrow 0$, which implies that $\{x_n\}$ is a Cauchy sequence.

Similarly, we can show that $\{y_n\}$ is also a Cauchy sequence. And therefore,by completeness of $X$, there exist $x$ *and* $y \in X$ such that $\lim_{n\to\infty} x_n = x$ and
$\lim_{n\to\infty} y_n = y$.
Now, we will prove the existence of coupled fixed point.

$$d(F(x, y), x) \precsim d(x_{n+1,} F(x, y)) + d(x_{n+1}, x).$$
$$\precsim d(F(x_n, y_n), F(x, y)) + d(x_{n+1}, x).$$
$$\precsim \Lambda (x_n)\, d(F(x_n, y_n), x_n) + \Xi(x)\, d(F(x, y), x,) .$$
$$\precsim \Lambda (x_0)\, d(x_{n+1}, x_n) + \Xi(x)\, d(F(x, y), x,) + d(x_{n+1}, x) .$$
$$\precsim \frac{\Lambda (x0) d(xn+1,xn) + d(xn+1,x)}{(1-\Xi(x)}$$

Proceeding limit $n\to\infty$ in above inequality, we get

$d(F(x, y), x) = 0$ which in turns implies that $F(x, y) = x$.

Similarly, we can show that $F(y, x) = y$.

Thus, F has coupled fixed point.

Finally, we will show that (x, y) is unique coupled fixed point of $F$.

Now, if $(x^*, y^*)$ is another coupled fixed point of $F$, then by applying (2.1), we have
$d(x, x^*) = d(F(x, y), F(x^*, y^*))$.
$$\precsim \Lambda(x)\, d(F(x,y), x) + \Xi(x^*)\, d(F(x^*,y^*), x^*) = 0.$$
Thus we have $x = x^*$. Similarly, we get $y = y^*$.
Therefore, $F$ has a unique coupled fixed point.

**Note:** If we take $\Lambda(x) = \Xi(x)$ in the above theorem then we obtain the following corollary:
**Corollary 2.2.** Let (X, d) be a complete complex valued metric space and $F$: X × X → X be a mapping.If there exists mapping $\Lambda$: X→ [0, 1) such that for all x, y ∈ X:

(i): $\Lambda(F(x,y)) \leq \Lambda(x)$ and $\Lambda(x) < 1$;
(ii): $d(F(x,y), F(u,v)) \precsim \Lambda(x)\, d(F(x,y), x) + \Lambda(u)\, d(F(u,v), u)$. (2.2)
Then F has a unique coupled fixed point.
Proof: we can prove the result by applying theorem 2.1 with $\Lambda(x) = \Xi(x)$.

**Example 2.3.** Let $X = C$ and define $d : X \times X \to C$ by $d(x, y) = i|x-y|$.
Then $(X, d)$ is a complete complex valued metric space. Consider the mapping
$F: X \times X \to X$ with $F(x, y) = i(x-y) - \frac{1}{5}$.
And $\Lambda$, $\Xi$: X→ [0, 1) defined by $\Lambda(x) = \frac{x+1}{2}$ and $\Xi(x) = \frac{x+1}{3}$.
Then F, $\Lambda$ and $\Xi$ satisfy the contraction
$d(F(x,y), F(u,v)) \precsim \Lambda(x)\, d(F(x,y), x) + \Xi(u)\, d(F(u,v), u)$.
As well as the conditions
$\Lambda(F(x,y)) \leq \Lambda(x)$ and $\Xi(F(x,y)) \leq \Xi(x)$;
$(\Lambda + \Xi)(x) < 1$;
Here $(-\frac{1}{5}, -\frac{1}{5})$ is the coupled fixed point of F.

## 3. DEDUCED RESULTS

We deduce the result of theorem 2.4 of [5] if we replace the mappings $\Lambda$, $\Xi$ by constants h and k in our main result.

## 4. COMPETING INTERESTS

The authors declare that they have no competing interests.

## 5. AUTHORS' CONTRIBUTIONS

All authors contributed equally and significantly to this research work. All authors read and approved the final manuscript.

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