Adjoint of a Lorentz Transformation

Gunjan Agrawal and Nisha Godani

Department of Mathematics, Faculty of Science

Dayalbagh Educational Institute (Deemed University) Dayalbagh, Agra, Uttar Pradesh, India

ABSTRACT

Lorentz transformations describe the relationship between space and time measurements, measured in two different inertial frames of reference in Special Theory of Relativity. These transformations are linear and preserve Lorentz inner product on the 4-dimensional Minkowski space, a mathematical setting which represents the spacetime of Special Theory of Relativity. In the present paper, defining adjoint of a linear operator on Minkowski space, characterizations for Lorentz transformations have been obtained. This provides a new approach for the study of Lorentz transformations. A method to compute the matrix associated with the adjoint directly from the matrix of a given linear operator has been developed. Further, studying properties of adjoint of a linear operator on Minkowski space, it has been obtained that its determinant is same as the determinant of the given linear operator. It has been observed that Lorentz transformations are analogues of unitary transformations in the context of finite dimensional Minkowski space.

Keywords: Lorentz Transformation, Minkowski space, Adjoint, Unitary Transformation

1. INTRODUCTION

Lorentz transformations, first described by Dutch Physicist Hendrik A. Lorentz in 1890, are pedagogically important transformations which relate the velocity components of an object observed in two inertial frames of reference in Special Theory of Relativity and have large number of derivations. The successful theory of these transformations has several applications in Thermodynamics, Plasma Physics etc. In 1988, T. Chang, D. G. Torr and D. R. Gagnon[1] modified Lorentz theory as a test theory of Special Relativity and further, T. Chang and D. G. Torr [2] studied dual properties of spacetime under an alternative Lorentz transformation. G. L. Light [4], in 2010, has clarified the Doppler violet/ red shifts within the domain of Special theory of relativity by the eigenvalues of Lorentz transformations. Further, in 2010, these transformations have been applied to Aether space [6].

Minkowski space named after the German mathematician Hermann Minkowski, is geometry of spacetime very different from the Euclidean geometry. Minkowski's conception of spacetime has made a great impact on the interpretation of quantum theory, cosmology etc. Relativistic quantum theories are equipped with a background of Minkowski spacetime. Cauchy's problems for harmonic maps, 2-dimensional quantum gravity, finite temperature quantum field theory etc. are defined on Minkowski space. Lorentz transformation describes rotation on Minkowski space. Einstein theory of special relativity is formulated in 4-dimensional Minkowski space. The present paper is focused on the study of Lorentz transformations in terms of adjoint. Sectionwise description of the paper is given below:

The paper begins with the necessary preliminaries in Section 2. In Section 3, defining adjoint of a linear operator on Minkowski space, its properties have been studied and characterizations for Lorentz transformations have been obtained. Finally, Section 4 concludes the paper.

2. LORENTZ TRANSFORMATION

The n-dimensional real vector space R^n with bilinear form $\langle , \rangle : R^n \times R^n \to R$, satisfying the following properties: (i) symmetric, i.e. $\langle x, y \rangle = \langle y, x \rangle$ for all x, $y \in R^n$ (ii) nondegenerate, i.e. if $\langle x, y \rangle = 0$ for all $x \in R^n$, then $y = 0$ and (iii) is of index 1, i.e. there exists a basis e_0, e_1, ..., e_{n-1} for R^n with

$$\langle e_i, e_j \rangle = \eta_{ij} = \begin{cases} 1 \text{ if } i=j=0 \\ -1 \text{ if } i=j=1,\ldots,n\text{-}1 \\ 0 \text{ if } i \neq j \end{cases}$$

is called the *n-dimensional Minkowski space*, denoted by M. The bilinear form $\langle , \rangle$ is called the *Lorentz inner product*, the matrix η_{ij} is known as the *Minkowski metric* and an element $x \in M$ is called *vector*. For $x = \sum_{i=0}^{n-1} x_i e_i$, the coordinate x_0 is called the time component and the coordinates $x_1, x_2, \ldots, x_{n-1}$ are called the spatial components of x relative to the basis e_0, e_1, ..., e_{n-1}. Then, the Lorentz inner product $\langle x, y \rangle$ of two events x and y is equal to $x_0 y_0 - \sum_{i=1}^{n-1} x_i y_i$, where $x = \sum_{i=0}^{n-1} x_i e_i$ and $y = \sum_{i=0}^{n-1} y_i e_i$. A vector $v \in M$ is said to be *lightlike*, *timelike* or *spacelike* vector according as $\langle v, v \rangle = 0$, >0 or <0. For nonzero lightlike vectors $v, w \in M$ $\langle v, w \rangle = 0$ iff $v = tw$, where $t \in R$. If $v \in M$ is a nonzero timelike vector such that $\langle v, w \rangle = 0$, then w is spacelike. Unlike usual inner product, Lorentz inner product is not positive definite. Also, usual inner product induces a norm on M and hence topology, while Lorentz inner product does not. Further, for $v, w \in M$, if $\langle a, v \rangle = \langle a, w \rangle$ for all $a \in M$, then $v = w$[5].

A linear operator U on a 4-dimensional Minkowski space M is said to be a Lorentz transformation if $\langle Uv, Uv \rangle = \langle v, v \rangle$, for all $v \in M$. It is well known that U is a Lorentz transformation iff

$\langle Uv, Uw\rangle = \langle v, w\rangle$, for all $v, w \in M$ which implies that $v \in M$ is lightlike, timelike or spacelike iff Uv is lightlike, timelike or spacelike. Further, a Lorentz transformation is bijective, maps orthonormal basis to orthonormal basis and the determinant of the associated matrix is ±1. Also, the eigenvalues for which the corresponding eigenvectors are non-lightlike, are ±1 and the product of those eigenvalues is 1 for which the corresponding eigenvectors are linearly independent and lightlike [5, 8].

3. ADJOINT

Adjoint of an operator on Hilbert space is well studied [7]. In this section, the notion of adjoint of a linear operator on M has been introduced and its properties have been studied. Further, characterizations for a Lorentz transformation have been obtained. Throughout this section, for $c \in M$, f_c denotes a linear functional on M defined by $f_c(x) = \langle x, c\rangle$ for all $x \in M$.

Definition 3.1: Let T be a linear operator on M. A linear operator T^*on M is called *adjoint* of T if $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x, y \in M$.

Proposition 3.2: *Let f be a linear functional on M. Then there exists a unique vector b ∈ M such that f =*f_b.

Proof: Let $\{e_0, e_1, \dots, e_{n-1}\}$ be an orthonormal basis for M. Set $b = a_0e_0 - \sum_1^{n-1} a_i\, e_i$, where $a_i = f(e_i)$, for each $i = 0, 1, \dots., n-1$. Then $b \in M$ and by the properties of Lorentz inner product, it follows that for $x \in M$, $\langle x, b\rangle = f(x)$. Further, to prove that b is unique, let $c \in M$ be such that $\langle x, b\rangle = \langle x, c\rangle$ for all $x \in M$. Then $b = c$. This completes the proof.

It is known that adjoint of a linear operator on a finite dimensional real inner product space V exists and is unique [3]. It has been found in the following proposition that a similar result holds for a linear operator on M.

Proposition 3.3: *Let T be a linear operator on M. Then adjoint* T^* *of T exists and is unique.*

Proof: To prove the existence of T^*, let $y \in M$. Then f_y is a linear functional on M. By Proposition 3.2, there exists $b \in M$ such that $f_y o T = f_b$ which implies that for $x \in M$, $\langle Tx, y\rangle = \langle x, b\rangle$. Define $T^*: M \to M$ such that $T^*y = b$. Then $\langle Tx, y\rangle = \langle x, T^*y\rangle$.

To prove that T^* is linear, let $y, z \in M$ and c and d be scalars. Then for $x \in M$, by the properties of Lorentz inner product, $\langle Tx, cy + dz\rangle = \langle x, T^*(cy + dz)\rangle$. $\langle Tx, cy + dz\rangle = \langle Tx, cy\rangle + \langle Tx, dz\rangle = c\langle Tx, y\rangle + d\langle Tx, z\rangle = \langle x, cT^*y\rangle + \langle x, dT^*z\rangle = \langle x, cT^*y + dT^*z\rangle$. Hence, $T^*(cy + dz) = cT^*y +$

dT^*z. To prove uniqueness, let T^* and T' be operators on M such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$ and $\langle Tx, y\rangle = \langle x, T'y\rangle$ for all $x, y \in M$. Then $\langle x, T^*y\rangle = \langle x, T'y\rangle$ for all $x, y \in M$, finally proves that $T^* = T'$.

3.4 Matrix of T^*: Let T be a linear operator on M such that $Tx = y$, for $x = (x_0, x_1, \ldots, x_{n-1})$, $y = (y_0, y_1, \ldots, y_{n-1})$ and $y_i = \sum_{j=0}^{n-1} c_{i,j}x_j$. Then for a linear functional *f* on *M*, by Proposition 3.2, there exists a unique vector $b = (b_0, b_1, \ldots, b_{n-1}) \in M$ such that $f_b(Tx) = \langle T(x), b\rangle$. By definition of Lorentz inner product, $f_b(Tx) = y_0b_0 - y_1b_1 - \ldots - y_{n-1}b_{n-1}$. Since each $y_i = \sum_{j=0}^{n-1} c_{i,j}x_j$, therefore

$$f_b(Tx) = b_0 \sum_{j=0}^{n-1} c_{0,j}x_j - \sum_{k=1}^{n-1} b_k \left(\sum_{j=0}^{n-1} c_{k,j}x_j\right) =$$

$$x_0\left(c_{0,0}b_0 - \sum_{k=1}^{n-1} c_{k,0}b_k\right) - \sum_{k=1}^{n-1} x_k \left(-c_{0,k}b_0 + \sum_{j=1}^{n-1} c_{j,k}b_j\right) = \langle x, T^*b\rangle.$$

Thus, the matrix of T^* with respect to the standard basis is $A^* = \begin{bmatrix} c_{0,0} & -c_{1,0} & -\ldots & -c_{n-1,0} \\ -c_{0,1} & c_{1,1} & \ldots & c_{n-1,1} \\ . & & & . \\ . & & & . \\ . & & & . \\ -c_{0,n-1} & c_{1,n-1} & \ldots & c_{n-1,n-1} \end{bmatrix}$

Hence the matrix A^* is obtained from the matrix of *T*, denoted by *A*, by multiplying the last (*n*-1) entries of first row and first column of *A* by -1 and then taking its transpose.

Example 3.5: Let T be a linear operator on a 3-dimensional Minkowski space defined as $T(x, y, z) = (x + y + z, x - y + z, x + y - z)$. Then $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$ and $A^* = \begin{bmatrix} 1 & -1 & -1 \\ -1 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix}$.

Remark 3.6: Like finite dimensional real inner product spaces, the adjoint of a matrix *A* on *M* is not equal to its transpose.

It is well known that if A and A^* be the matrices associated with linear operators T and T^* on a finite dimensional real inner product space, then detA^* = det$A$ [3]. It has been obtained in the following proposition that a similar result holds for a linear operator on *M*.

Proposition 3.7: Let A be the matrix associated with a linear operator T on M and A^* be the matrix associated with T^*. Then detA^* = det$A$.

Proof: Let A be an $n\times n$ matrix, $n\in N$. Then det$A$ = $\sum_{\sigma}(\text{sgn }\sigma)A(1,\sigma 1)A(2,\sigma 2)...A(n,\sigma n)$, where σ is a permutation of degree n and $A(j,\sigma j)$ is the $(j,\sigma j)$th entry of the matrix A, $1 \leq j \leq n$ [4]. As obtained in Article 3.4 above, $A^*(j,\sigma j) = -A(\sigma j, j)$ if either j = 1 or σj = 1, but $j \neq \sigma j$; otherwise $A^*(j,\sigma j) = A(\sigma j, j)$. Thus det$A^*$ $= \sum_{\sigma}(\text{sgn }\sigma)A^*(1,\sigma 1)A^*(2,\sigma 2)...A^*(n,\sigma n) =$ $\sum_{\sigma}(\text{sgn }\sigma)A(\sigma 1,1)A(\sigma 2,2)...A(\sigma n,n) = \sum_{\sigma}(\text{sgn }\sigma)A(1,\sigma^{-1}1)A(2,\sigma^{-1}2)...A(n,\sigma^{-1}n) = \det A.$

Proposition 3.8: Let M be a Minkowski space. Let T and U be linear operators on M and c be a scalar. Then

$(T + U)^* = T^* + U^*$

$(cT)^* = cT^*$

$(TU)^* = U^*T^*$

$(T^*)^* = T$

Proof: (i) For $x, y \in M$, $\langle x, (T + U)^*y\rangle = \langle (T + U)x, y\rangle$, by the definition of adjoint. Further, by the properties of Lorentz inner product and definition of adjoint $\langle (T + U)x, y\rangle = \langle Tx + Ux, y\rangle =$ $\langle Tx, y\rangle + \langle Ux, y\rangle = \langle x, T^*y\rangle + \langle x, U^*y\rangle = \langle x, (T^* + U^*)y\rangle$. This implies that $\langle x, (T + U)^*y\rangle =$ $\langle x, (T^* + U^*)y\rangle$ for all $x, y \in M$. Hence $(T + U)^* = T^* + U^*$.

(ii) Similar to (i) above.

(iii) By the definition of adjoint, $\langle x, (TU)^*y\rangle = \langle (TU)x, y\rangle = \langle Ux, T^*y\rangle = \langle x, U^*T^*y\rangle$ for $x, y \in M$. This implies that $\langle x, (TU)^*y\rangle = \langle x, U^*T^*y\rangle$ for all $x, y \in M$. Hence, $(TU)^* = U^*T^*$.

(iv) For $x, y \in M$, $\langle x, (T^*)^*y\rangle = \langle T^*x, y\rangle = \langle x, Ty\rangle$, by the definition of adjoint. Hence $(T^*)^* = T$.

Proposition 3.9: *Let U be a linear operator on M. Then the following are equivalent:*

(i) *U is Lorentz transformation.*

(ii) $U^*U = I$

(iii) $UU^* = I$

Proof: To prove (i) implies (ii). It is known that if U is a Lorentz transformation, then $\langle Ux, Uy \rangle = \langle x, y \rangle$ for all $x, y \in M$. By nondegeneracy of Lorentz inner product, $\langle x, U^*Uy \rangle = \langle x, y \rangle$ implies $U^*U = I$. To prove (ii) implies (iii), let $U^*U = I$. Then $A^*A = I$, where A is the matrix associated with U. By the property of determinant and by Proposition 3.7, det $A \neq 0$. Therefore A is invertible and $A^{-1} = A^*$. This implies that $AA^* = I$. Hence $UU^* = I$. To prove (iii) implies (i), let $UU^* = I$. Then $U^*U = I$ by the same argument as in the proof of (ii) implies (iii). Now, $\langle Ux, Ux \rangle = \langle x, U^*Ux \rangle = \langle x, Ix \rangle = \langle x, x \rangle$. This completes the proof.

Remark 3.10: Proposition 3.9 shows that Lorentz transformations on Minkowski space are analogues of unitary operators on finite dimensional real inner product space.

Proposition 3.11: *Let U be a Lorentz transformation on M. Then U^* is a Lorentz transformation.*

Proof: By the definition of adjoint, $\langle U^*x, U^*x \rangle = \langle x, UU^*x \rangle$ for all $x \in M$. By Proposition 3.9, $\langle x, UU^*x \rangle = \langle x, x \rangle$. Hence U^* is a Lorentz transformation.

4. CONCLUSION

In this paper, generalizing the well known notion of the adjoint of a linear operator from finite dimensional real inner product space to the *n*-dimensional Minkowski space, necessary and suffient conditions have been obtained for Lorentz transformations. It has been concluded that Lorentz transformations are analogue of unitary operators on finite dimensional real inner product space. Thus, the change in inner product has provided a simple and revealing structure theory for operators on Minkowski space resulting in an interesting exploration of an important area of Mathematics.

REFERENCES

[1] Chang T, Torr D G, Gagnon D R. A Modified Lorentz Theory as a Test Theory of Special Relativity, Found. Phys. Letters 1988.
[2] Chang T, Torr D G. Dual properties of spacetime under an alternative Lorentz transformation, Found. Phys. Letters 1988.
[3] Hoffman K, Kunze R. Linear Algebra (PHI Learning Pvt. Ltd., New Delhi 2011).
[4] Light G L. A Distinction between the Lorentz Transformation Eigenvalues and Factor, Adv. Studies Theor. Phys. 2010.
[5] Naber G L. The Geometry of Minkowski Spacetime (Springer-Verlag, New York 1992).
[6] Ranjan, Conard. DSSU Relativity: The Lorentz Transformations Applied to Aether-Space, Phys. Essay 2010.
[7] Simmons G F. Introduction to Topology and Modern Analysis, (Tata Mcgraw Hill, New Delhi 2006).
[8] Miguel A′ngel Javaloyes Victoria, Miguel S´anchez Caja, An introduction to Lorentzian Geometry and its applications, Webs.um.es/majava/charlas/cursodeLorentz.pdf 2010.