Application of Laplace-Differential Transform Method to Solve Non-linear PDEs with Boundary Conditions

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Abstract: **In this paper, a novel method called Laplace-differential transform method (LDTM) is presented to obtain an approximate analytical solution for strong nonlinear boundary value problems associated in engineering phenomena. It is determined that the method works very well for the wide range of parameters and an excellent agreement is demonstrated and discussed between the approximate solution and the exact one in three examples. The most significant features of this method are its simplicity and excellent accuracy for the finding. Also, the results reveal that this technique is very effective and convenient for solving complex nonlinearities.**

Keywords: **LDTM, Non-linear PDEs, Boundary conditions.**

1. INTRODUCTION

A nonlinear phenomenon appears in a wide variety of scientific and engineering applications such as chemical physics, fluid dynamics, chemical kinetics, solid state physics and transport in porous medium etc. In the analysis of mathematical models, those techniques are required for providing solutions conforming the physical reality and obtain the exact and approximate solutions. In 1978, a Russian mathematician G.E. Pukhov proposed Differential transform method (DTM) and started from computational structure to solve differential equations by Taylor's transformation. But in 1986, J. K. Zhou [7] has proposed the algorithm and main application of DTM with linear and non-linear initial value problems in electric circuit analysis. Alquran *et* al. [1] have used the combination of Laplace transform method and the DTM for solving non-homogeneous linear PDEs with variable coefficients. The researchers found that the LDTM requires less computational work compared to DTM. Hassan and Erturk [3] have studied the higher order boundary value problems and discussed the higher order series solution for linear and non-linear differential equations. Khan *et* al. [4] have given Laplace decomposition method to obtain the approximate solution of non-linear coupled PDEs and found that the Laplace decomposition method and Adomian decomposition method both can be used alternatively for the solution of higher order initial value problems. Mishra and Nagar [5] have applied a combined form of LTM with Homotopy perturbation method which is called He-Laplace method for solving linear and non-linear PDEs and found that the technique is capable to reduce the volume of computational work as compared to Adomian polynomials. Ravi Kanth and Aruna [6] have extended the DTM to solve the linear and non-linear Klein-Gordon equations and confirmed that the proposed technique provides its computational effectiveness and accuracy.

In this paper, we use a coupling of the Laplace transform method and the DTM for solving nonlinear PDEs with boundary conditions. The goodness of this method is its capability of combining two strongest methods for finding fast convergent series solution of PDEs. To the best of our knowledge no such try has been made to combine LTM and DTM for solving non-linear boundary value problems. This paper considers the effectiveness of the Laplace-differential transform method for solving nonlinear equations. In this paper, the definition of DTM is presented in Section 2. Section 3, contains basic idea of LDTM. Section 4, contains applications of LDTM. The conclusions are included in last Section.

2. DIFFERENTIAL TRANSFORMATION METHOD

The one variable differential transform [2] of a function $u(x,t)$ is defined as

$$U_k(t) = \frac{1}{k!}\left[\frac{\partial^k u(x,t)}{\partial x^k}\right]_{x=x_0} \quad ; k \geq 0 \tag{1}$$

where $u(x,t)$ is the original function and $U_k(t)$ is the transformed function. The inverse of one variable differential transform of $U_k(t)$ is defined as:

$$u(x,t) = \sum_{k=0}^{\infty} U_k(t)(x - x_0)^k, \tag{2}$$

where x_0 is the initial point for the given condition. Then the function $u(x,t)$ can be written as

$$u(x,t)=\sum_{k=0}^{\infty}U_k(t)x^k. \quad (3)$$

3. BASIC IDEA OF LDTM

To illustrate the basic idea of Laplace differential transform method [1], we consider the general form of inhomogeneous PDEs with variable or constant coefficients

$$£[u(x,t)]+\Re[u(x,t)]=f(x,t),\ x\in R,\ t\in R^+, \quad (4)$$

subject to the initial conditions

$$u(x,0)=g_1(x), \quad u_t(x,0)=g_2(x), \quad (5)$$

and the Dirichlet boundary conditions

$$u(0,t)=h_1(t), u(1,t)=h_2(t), \quad (6)$$

or the Neumann boundary conditions

$$u(0,t)=h_1(t), u_x(1,t)=h_3(t), \quad (7)$$

where $£[.]$ is linear operator w.r.to 't', $\Re$ [.] is remaining operator and *f* is a known analytical function.

First, we take the Laplace transform on both sides of equation (4), w.r.to '*t*', and we get

$$\mathrm{L}[£[u(x,t)]]+\mathrm{L}[\Re[u(x,t)]]=\mathrm{L}[f(x,t)]. \quad (8)$$

By using initial conditions from equation (5), we get

$$\bar{\mathrm{u}}(x,s)+\mathrm{L}[\Re[u(x,t)]]=\bar{f}(x,s), \quad (9)$$

where $\bar{\mathrm{u}}(x,s)$ and $\bar{f}(x,s)$ are the Laplace transform of $\mathrm{u}(x,t)$ and $f(x,t)$ respectively.

Afterwards, we apply differential transform method on the equation (9) w. r. to '*x*', and we get

$$\overline{\mathrm{U}}_k(s)+\mathrm{L}[\Re[U_k(t)]]=\overline{F}_k(s), \quad (10)$$

where $\overline{\mathrm{U}}_k(s)$ and $\overline{\mathrm{F}}_k(s)$ are the differential transform of $\bar{\mathrm{u}}(x,s)$ and $\bar{f}(x,s)$ respectively. In the next step, we apply inverse Laplace transform on both sides of the equation (10) w. r. to '*s*', and then we get

$$\mathrm{L}^{-1}\left[\overline{U}_k(s)\right]+\mathrm{L}^{-1}\mathrm{L}\left[\Re[U_k(t)]\right]=\mathrm{L}^{-1}\left[\overline{F}_k(s)\right],$$

or

$$U_k(t)+\Re[U_k(t)]=F_k(t). \quad (11)$$

Now, apply the differential transform method on the given Dirihlet and Neumann boundary conditions (6) and (7), we get

$$U_0(t)=h_1(t). \quad (12)$$

Let us assume

$$U_1(t)=a\,q(t). \quad (13)$$

By the definition of DTM [3], we take

$$u(1,t)=\sum_{i=0}^{\infty}U_i(t),\ u_x(1,t)=\sum_{i=0}^{\infty}i\,U_i(t). \quad (14)$$

By equation (14), we calculate the value of '*a*'.
Now by the above equations (12) and (13) in (11), the closed form series solution can be written as

$$u(x,t)=\sum_{k=0}^{\infty}U_k(t)x^k.$$

4. NUMERICAL EXAMPLES

To illustrate the applicability of LDTM, we have applied it to non-linear PDEs which are homogeneous as well as non-homogeneous. This method is capable to combine two methods for obtaining exact solutions.

Example 4.1: Consider the following homogeneous non-linear PDE [5]

$$\frac{\partial u}{\partial t}-u\frac{\partial u}{\partial x}-\frac{\partial^2 u}{\partial x^2}=0, \quad (15)$$

subject to the initial conditions

$$u(x,0)=1-x, \quad (16)$$

and the Dirichlet boundary conditions

$$u(0,t)=\frac{1}{1+t}, u(1,t)=0. \quad (17)$$

According to the Laplace differential transform method, firstly we applying the Laplace transformation w. r. to 't' on eqn. (15), we get

$$sL[u(x,t)]-u(x,0)=L\left[\frac{\partial^2 u}{\partial x^2}+u\frac{\partial u}{\partial x}\right]. \tag{18}$$

By using initial conditions from equation (16), we get

$$L[u(x,t)]=\frac{1-x}{s}+\frac{1}{s}L\left[\frac{\partial^2 u}{\partial x^2}+u\frac{\partial u}{\partial x}\right]. \tag{19}$$

Here, we applying the inverse Laplace transformation w. r. to '*s*', on both sides:

$$u(x,t)=1-x+L^{-1}\left[\frac{1}{s}L\left[\frac{\partial^2 u}{\partial x^2}+u\frac{\partial u}{\partial x}\right]\right]. \tag{20}$$

Now applying the Differential transformation method on equations (17) and (20) with respect to space variable '*x*', we get

$$U_k(t)=\delta(k,t)-\delta(k-1,t)+L^{-1}\left[\frac{1}{s}L[(k+2)(k+1)U_{k+2}(t)]\right]$$
$$+L^{-1}\left[\frac{1}{s}L\left[\sum_{r=0}^{k}(r+1)U_{r+1}(t)U_{k-r}(t)\right]\right]; \tag{21}$$

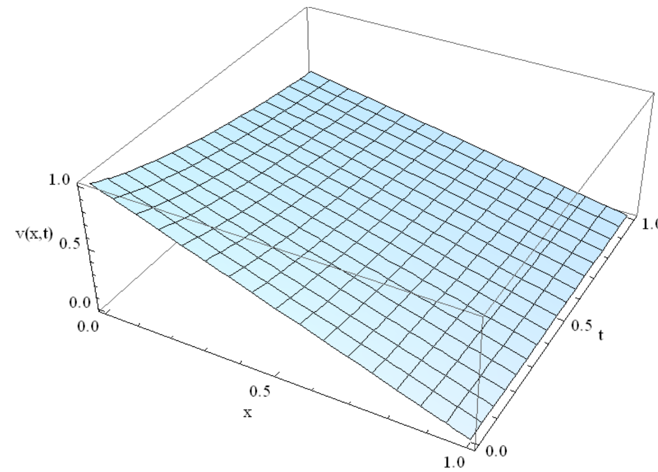
where, $\delta(k,t)=\begin{cases}1; & k=0\\ 0; & k\neq 0\end{cases}$ $\qquad U_0(t)=\frac{1}{1+t}.$ (22)

And let us assume

$$U_1(t)=\frac{a}{1+t}. \tag{23}$$

Substituting (22) and (23) into (21) and by straightforward iterative steps, we obtain

$$U_2(t)=-\frac{(a+1)}{2(1+t)^2}, \qquad U_3(t)=\frac{a+1}{3!(1+t)^2}\left(\frac{1}{1+t}-a\right),$$
$$U_4(t)=\frac{a+1}{4!(1+t)^3}\left[2(2a+1)-\frac{1}{1+t}\right],\ldots \tag{24}$$



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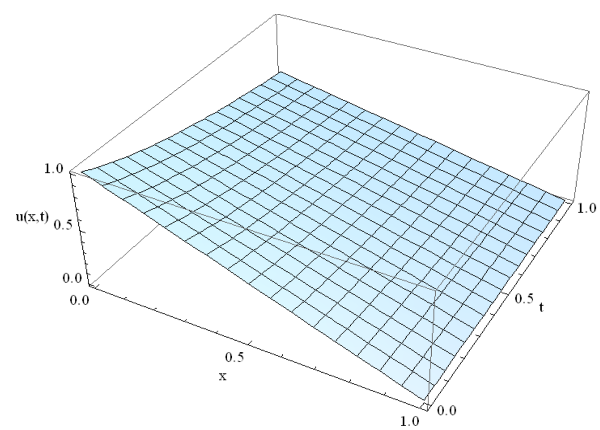


(b)

Fig. 1: The behavior of the (a) Exact solution, (b) LDTM solution, w.r.to x and t are obtained for the Example 4.1.

From equation (14), we take

$$u(1,t)=\sum_{i=0}^{\infty}U_i(t), \tag{25}$$

and, we get

$a=-1.$

Therefore,

$$U_1(t)=\frac{-1}{1+t}, U_2(t)=0, U_3(t)=0, U_4(t)=0,\ldots \tag{26}$$

Proceeding in this manner the components $U_k(t),\ k\geq 0$ of the DTM can be completely obtained. The differential transformation of $U_k(t)$ will give the series solution

$$u(x,t)=\sum_{k=0}^{\infty}U_k(t)x^k. \quad (27)$$

When we substitute all values of $U_k(t)$ from equation (22) and (26) into equation (27), the series solution can be formed as

$$u(x,t)=\frac{1-x}{1+t}. \quad (28)$$

which is the exact solution.

Example 4.2: The non-homogeneous non-linear PDE [6]

$$\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}+u^2=6xt\left(x^2-t^2\right)+x^6t^6, \quad (29)$$

subject to the initial conditions

$$u(x,0)=0,\ \ u_t(x,0)=0, \quad (30)$$

and the Neumann boundary conditions

$$u(0,t)=0,\ \ u_x(1,t)=3t^3. \quad (31)$$

According to the Laplace differential transform method, firstly we applying the Laplace transformation with respect to 't' on eqn. (29), we get

$$s^2L[u(x,t)]-su(x,0)-u_t(x,0)=\frac{6x^3}{s^2}-\frac{36x}{s^4}$$

$$+\frac{6!x^6}{s^7}+L\left[\frac{\partial^2 u(x,t)}{\partial x^2}-u^2(x,t)\right]. \quad (32)$$

By using initial conditions from equation (30), we get

$$L[u(x,t)]=\frac{6x^3}{s^4}-\frac{36x}{s^6}+\frac{6!x^6}{s^9}+L\left[\frac{\partial^2 u(x,t)}{\partial x^2}-u^2(x,t)\right].$$

Here, we applying the Inverse Laplace transformation w.r.t. 's' on both sides, and we get

$$u(x,t)=x^3t^3-\frac{3xt^5}{10}+\frac{x^6t^8}{56}$$

$$+L^{-1}\left[\frac{1}{s^2}L\left[\frac{\partial^2 u(x,t)}{\partial x^2}-u^2(x,t)\right]\right]. \quad (33)$$

Now applying the Differential transformation method on equations (31) and (33) with respect to space variable '*x*', we get

$$U_k(t)=t^3\delta(k-3,t)-\frac{3}{10}t^5\delta(k-1,t)+\frac{t^8}{56}\delta(k-6,t)$$

$$+L^{-1}\left[\frac{1}{s^2}L\left[(k+2)(k+1)U_{k+2}(t)-\sum_{r=0}^{k}U_r(t)U_{k-r}(t)\right]\right], \quad (34)$$

$$U_0(t)=0. \quad (35)$$

And let us assume

$$U_1(t)=at^4. \quad (36)$$

Substituting (35) and (36) into (34) and by straightforward iterative steps, we obtain

$$U_2(t)=0,\ U_3(t)=2at^2+t^3,\ U_4(t)=\frac{a^2t^8}{12},$$

$$U_5(t)=\frac{a}{5},\ U_6(t)=\frac{13a^2t^6}{45}+\frac{at^7}{15},\ldots \quad (37)$$

From equation (14), we take

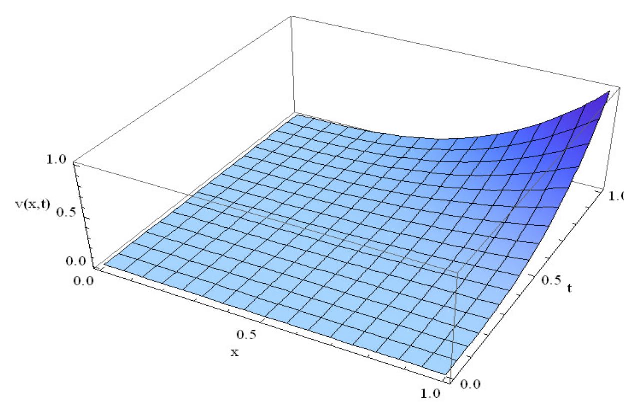
$$u_x(1,t)=\sum_{i=0}^{\infty}iU_i(t).$$

and, we get

$$a=0.$$

Therefore,

$$U_1(t)=0,U_2(t)=0,U_3(t)=t^3,U_4(t)=0,\ldots \quad (38)$$



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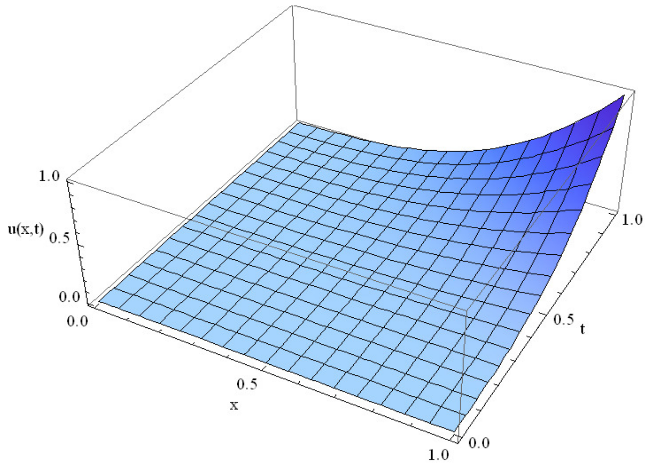


(b)

Fig. 2: The behavior of the (a) Exact solution, (b) LDTM solution, w.r.to x and t are obtained for the Example 4.2.

Proceeding in this manner the components $U_k(t),\ k \geq 0$ of the DTM can be completely obtained. The differential transformation of $U_k(t)$ will give the series solution

$$u(x,t) = \sum_{k=0}^{\infty} U_k(t) x^k. \tag{39}$$

When we substitute all values of $U_k(t)$ from equation (35) and (38) into equation (39), the series solution can be formed as

$u(x,t) = x^3 t^3.$

which is the exact solution.

Example 4.3: Consider the following non-homogeneous non-linear PDE [5]

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2 = 2x + t^4, \tag{40}$$

subject to the initial conditions

$$u(x,0) = 0,\ u_t(x,0) = a, \tag{41}$$

and the Neumann boundary conditions

$$u(0,t) = at, u_x(1,t) = t^2. \tag{42}$$

According to the Laplace differential transform method, firstly we applying the Laplace transformation with respect to ‘t’ on eqn. (40), we get

$$s^2 L[u(x,t)] - su(x,0) - u_t(x,0) = \frac{2x}{s}$$

$$+ \frac{4!}{s^5} - L\left[\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2\right]. \tag{43}$$

By using initial conditions from equation (41), we get

$$L[u(x,t)] = \frac{a}{s^2} + \frac{2x}{s^3} + \frac{4!}{s^7} - \frac{1}{s^2} L\left[\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2\right]. \tag{44}$$

Here, we applying the inverse Laplace transformation with respect to ‘s’, on both sides:

$$u(x,t) = at + xt^2 + \frac{t^6}{30} - L^{-1}\left[\frac{1}{s^2} L\left[\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2\right]\right]. \tag{45}$$

Now applying the Differential transformation method on equations (42) and (45) with respect to space variable ‘x’, we get

$$U_k(t) = at\delta(k,t) + t^2 \delta(k-1,t) + \frac{t^6}{30}\delta(k,t)$$
$$- L^{-1}\left[\frac{1}{s^2} L[(k+2)(k+1)U_{k+2}(t)]\right]$$

$$-L^{-1}\left[\frac{1}{s^2} L\left[\sum_{r=0}^{k} (r+1)(k-r+1) U_{r+1}(t) U_{k-r+1}(t)\right]\right], \tag{46}$$

$$U_0(t) = at. \tag{47}$$

And let us assume

$$U_1(t) = bt^2. \tag{48}$$

Substituting (47) and (48) into (46) and by straightforward iterative steps, we obtain

$$U_2(t) = \frac{(1-b^2)t^4}{2},\ U_3(t) = \frac{(1-b)[b(1+b)t^6 - 1]}{3},$$
$$U_4(t) = \frac{(1-b)t^2[6(1+b) + 2b\{1 - b(b+1)t^6\} - (1+b)t^6]}{12},$$
$$\ldots \tag{49}$$

From equation (14), we take

$$u_x(1,t)=\sum_{i=0}^{\infty} iU_i(t),$$

and, we get

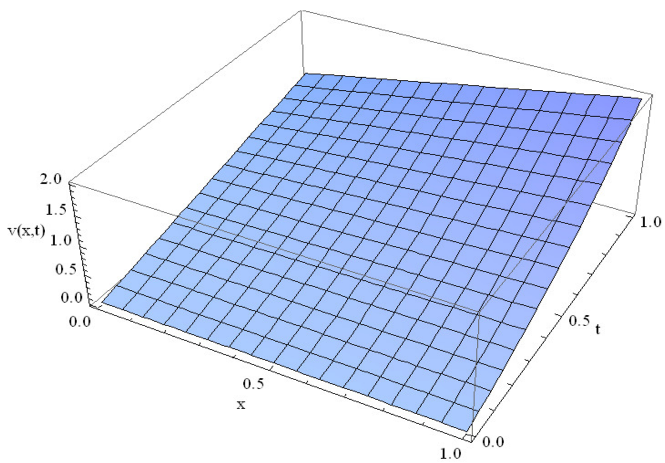
$b=1.$

Therefore,

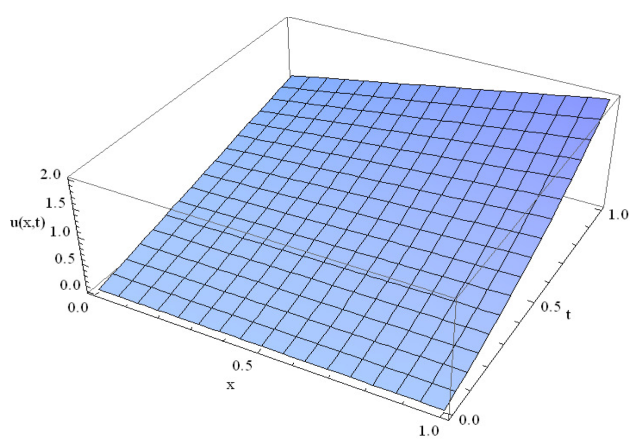
$$U_1(t)=t^2, U_2(t)=0, U_3(t)=0, U_4(t)=0,... \qquad (50)$$

Proceeding in this manner the components $U_k(t),\ k\geq 0$ of the DTM can be completely obtained. The differential transformation of $U_k(t)$ will give the series solution

$$u(x,t)=\sum_{k=0}^{\infty} U_k(t)x^k. \qquad (51)$$



(a)



(b)

Fig. 3: The behavior of the (a) Exact solution, (b) LDTM solution, w.r.to x and t are obtained for the Example 4.3.

When we substitute all values of $U_k(t)$ from equation (47) and (50) into equation (51), the series solution can be formed as

$$u(x,t)=at+xt^2.$$

which is the exact solution.

5. CONCLUSION

In this paper, the LDTM has been successfully applied to find the exact solution of homogeneous and non-homogeneous non-linear PDEs with boundary conditions. The aim of this paper is to describe that the LDTM is gives more reliable and reasonable solution of non-linear PDEs and it is easy to use. The proposed method requires less computational work compared with the DTM. The LDTM solution can be calculated easily in short time and the graphs were performed by using mathematica-8.

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