On Simultaneous Variation of Some Cosmological Parameters

S K J Pacif[1], Abdussattar[2]

[1]*Department of Mathematics, Manipal University, Jaipur, Jaipur, Rajasthan, India*
[2]*Department of mathematics, Faculty of Science, Banaras Hindu University, Varanasi, India*

Abstract: **Einstein field equations with time dependent G and Λ have been considered in such a way which conserves the energy momentum tensor of matter. Coupling of different cosmological parameters is discussed in a general gravitational situation. It is found that the ansatz $\rho \propto \theta^2$, which leads to suitable cosmological models with variable G and Λ in Robertson-Walker geometry with $k = +1$ and in Bianchi type-I space-time, is incompatible in Bianchi type-V space-time.**

Keywords: **Variable cosmic parameters, Anisotropic, homogeneous cosmological models, Einstein field equations with variable Λ and G.**

1. INTRODUCTION

In an evolving universe, the time variation of physically relevant parameters such as the energy density ρ, the volume expansion scalar θ (hence the Hubble parameter $H(= \theta/3)$ and the deceleration parameter $q = -1 - (\dot{H}/H^2)$) and the fluid shear scalar σ are important. In view of the constancy of both, the relative dynamical importance of the energy density and the relative dynamical importance of the rate of fluid shear observed in many special exact cosmological solutions of Einstein's field equations, Collins [1] established a theorem that holds in more general situation. The theorem states that "For any space-time in which the matter content consists of a perfect fluid whose equation of state is $p = (\gamma - 1)\rho$ (where ρ is the energy density, p is the isotropic pressure and γ is a constant) and whose flow vector field forms an expanding geodesic and hypersurface-orthogonal congruence, then, if the cosmological constant is zero,
$\rho \propto \theta^2 \Rightarrow \sigma^2 \propto \theta^2$ and $R^* \propto \theta^2$

where θ and σ are respectively the volume expansion and the rate of shear of the fluid congruence, and R^* is the Ricci scalar curvature of the hypersurfaces orthogonal to the flow. In these space-times, $R^* \leq 0$." A result which however was not noticed by Collins is that, in these space-times the deceleration parameter q comes out to be a constant, but the converse holds only if $\sigma^2 \propto \theta^2$. Another result, which can be seen in this context, is that, when the isotropic pressure p is supplemented by the bulk viscous pressure $-\xi\theta$, where ξ is the bulk viscosity term satisfying $\xi = \xi_0 \rho^{\frac{1}{2}}$ is the coefficient of bulk viscosity, then the above results also hold good.

Einstein's cosmological constant Λ and the Newton's gravitational constant G which were earlier treated as true constants are no longer regarded as constants in cosmology now a days. The existence of Λ is favoured by the recent supernovae Ia observations [2] and which is also consistent with the recent anisotropy measurements of the cosmic microwave background (CMB) made by the WMAP experiment [3]. However, there is a fundamental problem related with the existence of Λ, which has been extensively discussed in the literature. Its value expected from the quantum field theory- calculations is about 120 orders of magnitude higher than that estimated from the observations. A phenomenological solution to this problem is suggested by considering Λ as a function of time, so that it was large in the early universe and got reduced with the expansion of the universe [4], [5]. Variation of Newton's gravitational parameter G was originally suggested by Dirac on the basis of his large numbers hypothesis [6]. As G couples geometry to matter, it is reasonable to consider $G = G(t)$ in an evolving universe when one considers $\Lambda = \Lambda(t)$. Many extensions of general relativity with $G = G(t)$ have been made ever since Dirac first considered the possibility of a variable G, though none of these theories has gained wide acceptance. However a new approach, which has been widely investigated in the past few years [7], [8], is appealing. It assumes the conservation of the energy-momentum tensor which consequently renders G and Λ as coupled field, similar to the case of G in original Brans-Dickie theory. This leaves Einstein's field equations formally unchanged. In this context, an approach is worth mentioning in which the scaling of $G(t)$ and $\Lambda(t)$ arise from an underlying renormalization group flow near an infrared attractive fixed point [9]. The resulting cosmology explains the high redshift SNe Ia and radio sources observations successfully [10]. It also describes the plank era reliably and provides a resolution to the horizon and flatness problems of the standard cosmology without any unnatural fine tuning of the parameters [11]. Gravitational theories with variable G have also been discussed in the context of induced gravity model where G is generated by means of a non-vanishing vacuum expectation value of a scalar field [12]. Different phenomenological cosmological models have been proposed in the past few years, which describe the evolution of these constants (better to call them parameters).

It is believed that the early universe was characterized by a highly irregular expansion mechanism which isotropized later [13]. The level of anisotropy left out by the era of decoupling is only about 10^{-5}, as is revealed by the CMB observations. It could be that whatever mechanism diminished Λ to its present value, could have also rendered the early highly anisotropic universe to the present smoothed out picture. We believe that the variation of all these parameters $\rho, \theta, \sigma, \Lambda$ and G should be linked together. Whatever physical processes are responsible for the evolution of one cosmological parameter should also be responsible for the evolution of others, implying that the different cosmological parameters are coupled together.

We shall keep ourselves limited to Einstein's field equations and to the parameters which appear explicitly therein. It would be worthwhile to mention that models with varying speed of light are recently being promoted. These are supported by claims, based on the measurements of distant quasar absorption spectra that the fine structure constant may have been smaller in the past. However, the speed of light c has a complex character having six different facets which come from many laws of physics that are a priori disconnected from the notion of light itself [14]. If it is the causal speed of which these theories are talking about, then one should not consider a varying c in general relativity unless the structure of the space-time metric is changed and reinterpreted. We consider $c = 1$ throughout our paper and discuss the coupling of the said parameters in a general gravitational situation. It is found that the variation of ρ as θ^2 is incompatible in a Bianchi type-V space-time with variable G and Λ.

2. FIELD EQUATIONS

The universe is assumed to be filled with a distribution of matter represented by the energy-momentum tensor of a perfect fluid

$$T_{ij} = (\rho + p)U_i U_j + p g_{ij} \text{ (in the units with } c = 1), \quad (1)$$

where ρ is the energy density of cosmic matter and p is its isotropic pressure. U_i is the fluid flow vector field which form an expanding geodesic and hypersurface- orthogonal congruence. The Einstein field equations with variable gravitational and cosmological constants are

$$R_{ij} - \frac{1}{2} R_k^k g_{ij}$$

$$= -8\pi G \left(T_{ij} - \frac{\Lambda}{8\pi G} g_{ij} \right). \quad (2)$$

With the above stated specializations of the fluid, the Raychaudhuri equation [15] reads as

$$\dot{\theta} + \frac{1}{3}\theta^2 + 8\pi G \left\{ \frac{1}{2}(\rho + 3p) - \frac{\Lambda}{8\pi G} \right\}$$

$$+2\sigma^2 = 0, \quad (3)$$

where $\theta \left(= U^i_{;i}\right)$ is the fluid's volume expansion and $\sigma^2 \left(= \frac{1}{2}\sigma_{ij}\sigma^{ij}\right)$ is the fluid shear. In view of the vanishing divergence of the Einstein tensor, equations (1) and (2) give

$$\dot{\rho} + (\rho + p)\theta + \rho\frac{\dot{G}}{G} + \frac{\dot{\Lambda}}{8\pi G} = 0. \quad (4)$$

where an overhead dot $(\cdot)$ denotes differentiation along the fluid flow vector U^i. We now assume as is common in cosmology that, the law of conservation of energy momentum tensor of matter holds $\left(T^{ij}_{;i} = 0\right)$, giving

$$\dot{\rho} + (\rho + p)\theta = 0. \quad (5)$$

By use of which, equation (4) yields

$$8\pi\rho\dot{G} + \dot{\Lambda} = 0, \quad (6)$$

which shows that the variation of ρ, G and Λ are coupled together. We also consider the γ-law equation of state in the form

$$p = (\gamma - 1)\rho, \quad (7)$$

where γ is a constant such that $1 \le \gamma \le 2$. Using (7) in (3) and (5) and then eliminating θ between these two, we obtain

$$8\pi G = \left(\frac{2}{3\gamma - 2}\right)\left\{\frac{1}{\gamma}\frac{\ddot{\rho}}{\rho^2} - \left(\frac{3\gamma+1}{3\gamma^2}\right)\frac{\dot{\rho}^2}{\rho^3} + \left(\frac{\Lambda - 2\sigma^2}{\rho}\right)\right\}. \quad (8)$$

Differentiating (8) and using in (6), we get

$$\frac{\dddot{\rho}}{\rho} - \frac{2}{3}\left(\frac{6\gamma+1}{\gamma}\right)\frac{\ddot{\rho}\dot{\rho}}{\rho^2} + \left(\frac{3\gamma+1}{\gamma}\right)\frac{\dot{\rho}^3}{\rho^3} + \frac{3}{2}\gamma^2\dot{\Lambda} - \gamma(\Lambda - 2\sigma^2)\frac{\dot{\rho}}{\rho} - \gamma(2\sigma^2)^{\cdot} = 0. \quad (9)$$

This can be rewritten with the help of (5) and (7) as

$$\frac{d}{dt}\left(\dot{\theta} + \frac{1}{3}\theta^2\right) + \gamma\theta\left(\dot{\theta} + \frac{1}{3}\theta^2\right)$$

$$-\frac{3}{2}\gamma\dot{\Lambda} + (\Lambda - 2\sigma^2)(-\gamma\theta)$$

$$+(2\sigma^2)^{\cdot} = 0. \quad (10)$$

A representative length l representing the volume behavior of the cosmic fluid is defined as

$$\frac{\dot{l}}{l} = \frac{1}{3}\theta. \quad (11)$$

Using equation (11) in equation (10) and multiplying l^2; equation (10) can be written as

$$\frac{d}{dt}\left(l^2\frac{\ddot{l}}{l}\right) + \left(\frac{3\gamma-2}{2}\right)\frac{d}{dt}(\dot{l}^2)$$

$$-\frac{\gamma}{2}\frac{d}{dt}(\Lambda l^2) + \frac{1}{3}\frac{d}{dt}(2\sigma^2 l^2)$$

$$+\left(\frac{6\gamma-4}{3}\right)\sigma^2 l\dot{l} = 0. \quad (12)$$

Integrating equation (12) and dividing by l^2; we get

$$\frac{\ddot{l}}{l} + \left(\frac{3\gamma-2}{2}\right)\frac{\dot{l}^2}{l^2} - \frac{\gamma}{2}\Lambda + \frac{1}{3}(2\sigma^2) + \left(\frac{6\gamma-4}{3}\right)\frac{1}{l^2}\int \sigma^2 l\, dl + \frac{\alpha}{l^2} = 0, \quad (13)$$

where α is an arbitrary constant of integration. This gives

$$\Lambda = \left(\frac{2}{\gamma}\right)\frac{\ddot{l}}{l} + \left(\frac{3\gamma-2}{2}\right)\frac{\dot{l}^2}{l^2} + \left(\frac{2\alpha}{\gamma}\right)\frac{1}{l^2} + \frac{4}{3\gamma}(\sigma^2) + \left(\frac{4(3\gamma-2)}{3\gamma}\right)\frac{1}{l^2}\int \sigma^2 l\, dl. \quad (14)$$

Equation (5) can be integrated with the help of (7) and (11) to give

$$\rho = Al^{-3\gamma}, \quad (15)$$

where $A > 0$ is an arbitrary constant of integration. In view of equations (5), (7), (11) and (14), equation (8) can be expressed as

$$G = \frac{1}{8\pi A}\left(\frac{2}{2-3\gamma}\right)$$

$$\left[l^{-3\gamma}\left(3\frac{\ddot{l}}{l} - \Lambda + 2\sigma^2\right)\right]. \quad (16)$$

Raychaudhuri equation (3) can be written in terms of l with the help of (11) as

$$3\frac{\ddot{l}}{l} = -\frac{8\pi G}{2}(\rho + 3p) + \Lambda - 2\sigma^2. \quad (17)$$

Multiplying this equation by $2l\dot{l}$ and making use of equations (5), (6) and (11), we get on integration

$$3\frac{\dot{l}^2}{l^2} + \frac{3k}{l^2} = 8\pi G\rho$$

$$+\Lambda - \frac{4}{l^2}\int \sigma^2 l\, dl, \quad (18)$$

where k is an arbitrary constant of integration, equation (18) is the generalized Friedmann equation in the presence of fluid shear and time dependent G and Λ, which provides the dynamics of the models. The constant k plays the role of curvature parameter in a homogeneous and isotropic FRW space-time ($\sigma = 0$). Equations (11), (14)-(16) describe the coupling of the parameters Λ, G, σ, ρ and θ. It is easy to see that these coupled equations do not form a closed system. However, if supplemented with one more assumption in a given Riemannian space-time representing geometry of the universe, these equations are sufficient to specify the model completely. Several cosmological models with variable G and Λ have been obtained in the last three decades with or without the presence of bulk viscosity in homogeneous isotropic or anisotropic background by assuming one more condition in the form $\rho = \rho_c$ [16], $\rho = constant$ [7]-[8], $(\rho + 3p)l^3 = constant$ [17], $q = constant$ [18]-[21], $H = A(l^{-n} + 1)$ [22]-[23], $\Lambda = \gamma l^{-2}$ [8], $\Lambda = 3\beta H^2$ [24]-[28], $\Lambda = \beta\frac{\ddot{l}}{l} + \frac{\dot{l}^2}{l^2} + 3\gamma l^{-2}$ [29], $G = At^n$ [7], [30]. In the next section we consider the phenomenological ansatz

$$\rho \propto \theta^2 \text{ or } \rho = \beta\theta^2, \quad (19)$$

where β is a constant of proportionality.

3. VARIATION OF ρ AS θ^2

Equation (19) with the help of equations (5) and (7) leads to

$$2\frac{\ddot{\rho}}{\rho} - 3\frac{\dot{\rho}^2}{\rho^2} = 0 \quad (20)$$

and

$$\frac{\dddot{\rho}}{\rho} - \frac{2}{3}\left(\frac{6\gamma+1}{\gamma}\right)\frac{\ddot{\rho}\dot{\rho}}{\rho^2}$$

$$+\left(\frac{3\gamma+1}{\gamma}\right)\frac{\dot{\rho}^3}{\rho^3} = 0. \quad (21)$$

The variation of the energy density ρ as given in equation (19) with (5) and (7) gives the time-variation of scale factor as

$$l = mt^{\frac{2}{3\gamma}}, m \text{ is a positive constant.} \quad (22)$$

which implies

$$\frac{2}{\gamma}\frac{\ddot{l}}{l} + \left(\frac{3\gamma-2}{\gamma}\right)\frac{\dot{l}^2}{l^2} = 0 \quad (23)$$

and consequently, equation (14) reduces to

$$\Lambda = \left(\frac{2\alpha}{\gamma}\right)\frac{1}{l^2} + \frac{4}{3\gamma}(\sigma^2)$$

$$+\left(\frac{4(3\gamma-2)}{3\gamma}\right)\frac{1}{l^2}\int \sigma^2 l\, dl. \quad (24)$$

The deceleration parameter q in this case comes out to be

$$q = \frac{3\gamma}{2} - 1. \tag{25}$$

It follows that when the cosmological constant Λ and the gravitational coupling constant G are taken to vary with time in space-times in which ρ varies as θ^2, the deceleration parameter q is always constant, but the converse may not be true. Moreover, the condition $\rho \propto \theta^2$ always leads to a big bang origin of the universe.

It may be noted that, though, the recent observations favour accelerating models at present but the possibility of decelerating models with small deceleration parameter are not completely ruled out. It has also been shown that the high redshift supernovae Ia can also be explained successfully in the decelerating models if one takes into account the absorption of light by the intergalactic metallic dust ejected from the supernovae explosions [5], [31].

Taking equation (20) as an additional assumption, Abdussattar and Vishwakarma [8] has obtained a FRW model with variable G and Λ and Vishwakarma [32] has discussed the simultaneous variation of Λ, G and σ^2 in Bianchi type-I space-time with variable G and Λ. In the next section we consider the coupling of these cosmological parameters in a Bianchi type-V space-time employing the ansatz (19).

4. BIANCHI TYPE-V MODEL

We consider a homogeneous and anisotropic Bianchi type-V space-time described by the line-element

$$ds^2 = -dt^2 + X^2(t)dx^2 + e^{2nx}\{Y^2(t)dy^2 + Z^2(t)dz^2\}. \tag{26}$$

For this metric the Einstein's field equations $G_{ij} = -8\pi G T_{ij}$ yield

$$\frac{\ddot{X}}{X} + \frac{\ddot{Y}}{Y} + \frac{\dot{X}\dot{Y}}{XY} - \frac{n^2}{X^2} = -8\pi G(t)(\gamma - 1)\rho + \Lambda(t), \tag{27}$$

$$\frac{\ddot{Y}}{Y} + \frac{\ddot{Z}}{Z} + \frac{\dot{Y}\dot{Z}}{YZ} - \frac{n^2}{X^2} = -8\pi G(t)(\gamma - 1)\rho + \Lambda(t), \tag{28}$$

$$\frac{\ddot{Z}}{Z} + \frac{\ddot{X}}{X} + \frac{\dot{Z}\dot{X}}{ZX} - \frac{n^2}{X^2} = -8\pi G(t)(\gamma - 1)\rho + \Lambda(t), \tag{29}$$

$$\frac{\dot{X}\dot{Y}}{XY} + \frac{\dot{Y}\dot{Z}}{YZ} + \frac{\dot{Z}\dot{X}}{ZX} - 3\frac{n^2}{X^2} = 8\pi G(t)\rho + \Lambda(t), \tag{30}$$

$$2\frac{\dot{X}}{X} - \frac{\dot{Y}}{Y} - \frac{\dot{Z}}{Z} = 0. \tag{31}$$

The volume expansion scalar θ and shear σ for the metric (26) are obtained as

$$\theta = \frac{\dot{X}}{X} + \frac{\dot{Y}}{Y} + \frac{\dot{Z}}{Z}, \tag{32}$$

$$\sigma^2 = \frac{1}{3}\left[\frac{\dot{X}^2}{X^2} + \frac{\dot{Y}^2}{Y^2} + \frac{\dot{Z}^2}{Z^2} - \left(\frac{\dot{X}\dot{Y}}{XY} + \frac{\dot{Y}\dot{Z}}{YZ} + \frac{\dot{Z}\dot{X}}{ZX}\right)\right]. \tag{33}$$

An average scale factor is obtained as $l = (XYZ)^{\frac{1}{3}}$. On integration equation (31) yields

$$X^2 = YZ. \tag{34}$$

Equations (27)-(29), with the help of equations (32) and (33) lead to

$$\frac{\dot{X}}{X} = \frac{\dot{l}}{l}, \tag{35}$$

$$\frac{\dot{Y}}{Y} = \frac{\dot{l}}{l} - \frac{k_1}{l^3}, \tag{36}$$

$$\frac{\dot{Z}}{Z} = \frac{\dot{l}}{l} + \frac{k_1}{l^3}, \tag{37}$$

where k_1 is a constants of integration. Equations (35)-(37) can be integrated further to yield

$$X(t) = m_1\, l(t)\,, \tag{38}$$

$$Y(t) = m_2\, l(t)\, exp\left[-k_1 \int \frac{dt}{l^3}\right], \tag{39}$$

$$Z(t) = m_3\, l(t)\, exp\left[k_1 \int \frac{dt}{l^3}\right], \tag{40}$$

where m_1, m_2, m_3 are arbitrary constants of integration satisfying $(m_1.m_2.m_3) = 1$. In view of (34) we get $m_1 = 1$ and $m_2 = m_3^{-1}$. Equation (33) for shear now reads as

$$\sigma^2 = \frac{k_1^2}{l^6}. \tag{41}$$

Equations (38)-(40) with the help of equation (22) give the values of the metric potentials as

$$X(t) = m_1\, m t^{2/3\gamma}, \tag{42}$$

$$Y(t) = m_2\, mt^{\frac{2}{3\gamma}}$$

$$exp\left\{-\frac{k_1}{m^3}\left(\frac{\gamma}{\gamma-2}\right)t^{\frac{\gamma-2}{\gamma}}\right\}, \tag{43}$$

$$Z(t) = m_3\, mt^{\frac{2}{3\gamma}}$$

$$exp\left\{\frac{k_1}{m^3}\left(\frac{\gamma}{\gamma-2}\right)t^{\frac{\gamma-2}{\gamma}}\right\}. \tag{44}$$

So that, with suitable transformation, the metric (26) takes the form

$$ds^2 = -dt^2 + m^2 t^{4/3\gamma}\begin{bmatrix} m_1^2 d\chi_1^2 \\ +m_2^2 e^{2n\chi_1 - 2\left(\frac{k_1}{m^3}\right)\left(\frac{\gamma}{\gamma-2}\right)t^{\frac{\gamma-2}{\gamma}}} d\chi_2^2 \\ +m_3^2 e^{2n\chi_1 + 2\left(\frac{k_1}{m^3}\right)\left(\frac{\gamma}{\gamma-2}\right)t^{\frac{\gamma-2}{\gamma}}} d\chi_3^2 \end{bmatrix}. \tag{45}$$

For the model (45) the different cosmological parameters are obtained as

$$\theta = \frac{2}{\gamma}\frac{1}{t}\ , \tag{46}$$

$$\sigma = \frac{k_1}{m^3}\frac{1}{t^{2/\gamma}}, \tag{47}$$

$$\rho = \frac{A}{m^{3\gamma}}\frac{1}{t^2}, \tag{48}$$

$$\Lambda = \left(\frac{2-\gamma}{\gamma}\right)\left(\frac{k_1^2}{m^6}\right)\frac{1}{t^{\frac{4}{\gamma}}}$$

$$+\left(\frac{2-3\gamma}{\gamma}\right)\left(\frac{n^2}{m_1^2 m^2}\right)\frac{1}{t^{4/3\gamma}}, \tag{49}$$

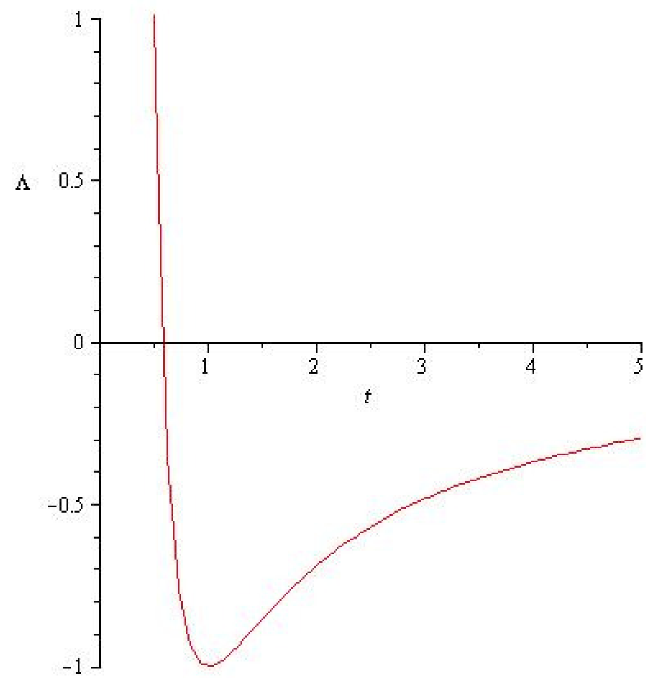
$$G = \left(\frac{m^{3\gamma}}{\pi A\gamma}\right)\left[\frac{1}{6\gamma} - \frac{1}{4}\left(\frac{k_1^2}{m^6}\right)\frac{1}{t^{(4/\gamma)-2}} - \frac{1}{4}\left(\frac{n^2}{m_1^2 m^2}\right)\frac{1}{t^{(4/3\gamma)-2}}\right]. \tag{50}$$

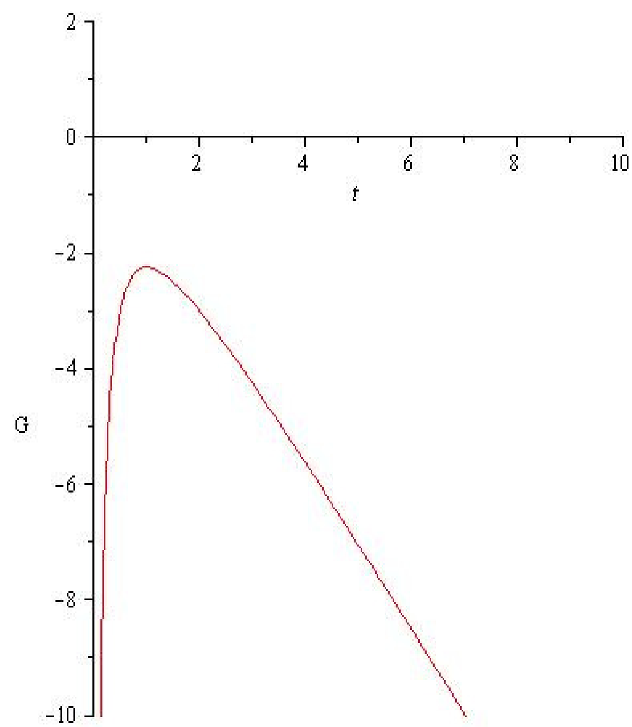
The anisotropy parameter

$\bar{A}\left(=\frac{1}{3}\sum_{i=1}^{3}\left(\frac{H_i - H}{H}\right)^2\right)$, where $H_1 = \frac{\dot{X}}{X}, H_2 = \frac{\dot{Y}}{Y}, \quad H_3 = \frac{\dot{Z}}{Z}$ are directional Hubble's factors, is obtained as

$$\bar{A} = \frac{3\gamma^2}{2}\left(\frac{k_1^2}{m^6}\right)\frac{1}{t^{(4/\gamma)-2}} \tag{51}$$

The anisotropy parameter $\bar{A}$ vanishes for large values of t. Therefore at late times, the model represents an isotropic universe. For the obtained model, the variation of Λ and G is depicted in the following figures.





We see that the gravitational coupling constant G remains negative throughout the evolution, whereas the cosmological constant Λ which is initially positive, becomes negative during the course of evolution and ultimately approaches to zero for large values of t. Thus, we conclude that the Bianchi type-V space-time with variable G and Λ is incompatible with the ansatz $\rho \propto \theta^2$.

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