# Chi Square Divergence Measure and Their Bounds

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*Abstract:* **Information and divergence measures are widely used in various applications of pattern recognition, signal processing and statistical applications. In this paper we shall study different methods to find chi-square divergence measures using well known convex functions and their compare to well established results which are well known in the literature of Information Theory. Their Relations are also establishing using some well-known inequalities in this research paper.**

***Keywords: - Chi-square divergence, f-divergence measure, Jenson-Shannon's divergenceetc.***

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## 1. INTRODUCTION

Let

$$\Gamma_n = \left\{ P = (p_1, p_2, \dots\dots p_n) \middle| p_i \geq 0, \sum_{i=1}^{n} p_i = 1 \right\}, n \geq 2$$

be the set of all complete finite discrete probability distributions. There are many information and divergence measures exists in the literature on information theory and statistics. Csiszar [2, 3] introduced a generalized measure of information using f-divergence functional

$$I_f(P,Q) = \sum_{i=1}^{n} q_i f\left(\frac{p_i}{q_i}\right) \quad (1.1)$$

Where $f : \mathbf{R}_+ \to \mathbf{R}_+$ is a convex function and $P, Q \in \Gamma_n$ . A class of divergence measures called f-divergences was introduced by Csiszar [2, 3]. Here we shall give some example of divergence measures in the category of Csiszar f-divergence measure.

- Relative Information[10] $D(P,Q) = \sum_{i=1}^{n} p_i \log \frac{p_i}{q_i}$ (1.2)
- Chi-square divergence[6]

$$\chi^2(P,Q) = \sum_{i=1}^{n} \frac{(p_i - q_i)^2}{q_i} = \sum_{i=1}^{n} \frac{p_i^2}{q_i} - 1 \quad (1.3)$$

- Relative J-S divergence[10]

$$F(P,Q) = \sum_{i=1}^{n} p_i \log\left(\frac{2p_i}{p_i + q_i}\right) \quad (1.4)$$

- Hellinger discrimination[5]

$$h(P,Q) = 1 - B(P,Q) = \frac{1}{2}\sum_{i=1}^{n}\left(\sqrt{p_i} - \sqrt{q_i}\right)^2 \quad (1.5)$$

where $B(P,Q) = \sum_{i=1}^{n} \sqrt{p_i q_i}$ is known as Bhattacharya divergence measure [1]

- J-divergence measure [4]

$$J(P,Q) = \sum_{i=1}^{n} (p_i - q_i) \log\left(\frac{p_i}{q_i}\right) \quad (1.6)$$

- Relative J-divergence measure [4]

$$J_R(P,Q) = \sum_{i=1}^{n} (p_i - q_i) \log\left(\frac{p_i + q_i}{2q_i}\right) \quad (1.7)$$

## 2. INFORMATION INEQUALITIES:

The following Inequalities have been established by Taneja [10]. Using their inequalities we have discussed particular cases for chi-square divergence measure.

**Theorem:** Let $f : \mathbf{R}_+ \to \mathbf{R}$ be the mapping which is normalized i.e. $f(1) = 0$. Suppose that

(i) $f$ is twice differentiable function on (r, R), $0 \leq r \leq 1 \leq R < \infty$ ( $f'$ and $f''$ denote the first and second derivatives of $f$ ),

(ii) there exist real constant $m, M$ such that $m < M$ and $m \leq x^{2-s} f''(x) \leq M \quad \forall x \in (r, R),\ s \in \mathbf{R}$

If $P,Q\in\Gamma_n$ are discrete probability distributions with $0<r\le\frac{p_i}{q_i}\le R<\infty$,

then $m\Phi_s(P,Q)\le I_f(P,Q)\le M\,\Phi_s(P,Q)$ (2.1)

and $$m\left[\eta_s(P,Q)-\Phi_s(P,Q)\right]\le I_f(P,Q)\le M\left[\eta_s(P,Q)-\Phi_s(P,Q)\right] \quad (2.2)$$

and $\eta_s(P,Q)=C_{\Phi'_s}\left(\frac{P^2}{Q},P\right)-I_{\Phi'_s}(P,Q)$ Where

$$\Phi_s(P,Q)=\begin{cases}{}^2K_s(P,Q), & s\ne 0,1\\ D(Q,P), & s=0\\ D(P,Q), & s=1\end{cases} \quad (2.3)$$

$$I_\rho(P,Q)=I_{f'}\left(\frac{P^2}{Q},P\right)-I_{f'}(P,Q)=\sum_{i=1}^{n}(p_i-q_i)f'\left(\frac{p_i}{q_i}\right) \quad (2.4)$$

and

$$\eta_s(P,Q)=C_{\Phi'_s}\left(\frac{P^2}{Q},P\right)-I_{\Phi'_s}(P,Q)$$

$$=\begin{cases}(s-1)^{-1}\sum_{i=1}^{n}(p_i-q_i)\left(\frac{p_i}{q_i}\right)^{s-1}, & s\ne 1\\ \sum_{i=1}^{n}(p_i-q_i)\log\left(\frac{p_i}{q_i}\right), & s=1\end{cases} \quad (2.5)$$

## 3. CHI-SQUARE DIVERGENCE MEASURE

In this section we shall establish chi-square divergence measure from following convex functions.

**Example:1** Let the convex function

$$\Phi(t)=t^2-1,\forall\ t>0 \quad (3.1)$$

$$\Phi(t)=2t$$

$$\Phi(t)=2\ge 0$$

Then we can findout the Csiszar's $\Phi$ -divergence measure

$$I_\phi(p,q)=\sum_{i=1}^{n}\frac{p_i^2}{q_i}-1=\chi^2(p,q) \quad (3.3)$$

$$I_{\phi'}(p,q)=2\sum_{i=1}q_i\left(\frac{p_i}{q_i}\right)=2\sum_{i=1}^{n}p_i=2$$

$$I_{\phi'}(\frac{p^2}{q},p)=2\sum_{i=1}^{n}p_i\left(\frac{p_i^2}{q_i}.\frac{1}{p_i}\right)=2\sum_{i=1}^{n}\frac{p_i^2}{q_i}-2+2$$

$$I_{\phi'}(\frac{p^2}{q},p)=2\sum_{i=1}^{n}p_i\left(\frac{p_i^2}{q_i}.\frac{1}{p_i}\right)=2\left(\sum_{i=1}^{n}\frac{p_i^2}{q_i}-1\right)+2$$

$$I_{\phi'}(\frac{p^2}{q},p)=2\chi^2(p,q)+2$$

***From (3.1)***

$$\phi'(1)(P_n-Q_n)\le I_\phi(p,q)-Q_n\phi(1)\le I_{\phi'}(\frac{p^2}{q},p)-I_{\phi'}(p,q)$$

$$2(P_n-Q_n)\le\chi^2(p,q)-0\le 2\chi^2(p,q)+2-2$$

$$2(P_n-Q_n)\le\chi^2(p,q)\le 2\chi^2(p,q)$$

***Two cases are created***

$$2(P_n-Q_n)\le\chi^2(p,q) \quad (3.2)$$

$$\chi^2(p,q)\le 2\chi^2(p,q)$$

$$0\le\chi^2(p,q)$$

***From (3.2)***

$$0\le I_\phi(p,q)-Q_n\Phi(\frac{P_n}{Q_n})\le I_{\phi'}(\frac{p^2}{q},p)-\frac{P_n}{Q_n}I_{\phi'}(p,q)$$

$$0\le\chi^2(p,q)-Q_n\left(\frac{P_n^2}{Q_n^2}-1\right)\le 2\chi^2(p,q)+2-\frac{2P_n}{Q_n}$$

***From (3.3)***

$$0\le I_\phi(p,q)\le I_{\phi'}(\frac{p^2}{q},p)-I_{\phi'}(p,q)$$

$$0\le\chi^2(p,q)\le 2\chi^2(p,q)+2-2$$

$$0\le\chi^2(p,q)\le 2\chi^2(p,q)$$

$$0\le\chi^2(p,q) \quad (3.3)$$

***Example: 3consider a convex function***

$$\Phi(t) = t(t-1), \ \forall\ t > 0$$

Function $\Phi$ is a convex function from Fig. 3.1and normalized i. e. f(1)=0 Then applying Csiszar's $\Phi$ -divergence measure

$$I_\phi(p,q) = \sum_{i=1}^{n} \frac{p_i^2}{q_i} - 1 = \chi^2(p,q)$$

$$I_{\phi'}(p,q) = \sum_{i=1}^{n} q_i \left( 2\frac{p_i}{q_i} - 1 \right)$$

$$I_{\phi'}(p,q) = 2\sum_{i=1}^{n} p_i - \sum_{i=1}^{n} q_i$$

$$I_{\phi'}(p,q) = 1$$

$$I_{\phi'}(\frac{p^2}{q}, p) = 2\chi^2(p,q) + 1$$

***From (3.1)***

$$\phi'(1)(P_n - Q_n) \le I_\phi(p,q) - Q_n\phi(1)$$

$$\le I_{\phi'}(\frac{p^2}{q}, p) - I_{\phi'}(p,q)$$

$$(P_n - Q_n) \le \chi^2(p,q) - Q_n \le 2\chi^2(p,q) + 1 - 1$$

$$(P_n - Q_n) \le \chi^2(p,q) - Q_n \le 2\chi^2(p,q)$$

***From (3.2)***

$$0 \le I_\phi(p,q) - Q_n\Phi(\frac{P_n}{Q_n})$$

$$\le I_{\phi'}(\frac{p^2}{q}, p) - \frac{P_n}{Q_n} I_{\phi'}(p,q)$$

$$0 \le \chi^2(p,q) - Q_n \frac{P_n}{Q_n}\left(\frac{P_n}{Q_n^2} - 1\right)$$

$$\le 2\chi^2(p,q) + 1 - \frac{P_n}{Q_n}$$

$$0 \le \chi^2(p,q) - P_n\left(\frac{P_n}{Q_n^2} - 1\right) \le 2\chi^2(p,q) + \frac{Q_n - P_n}{Q_n}$$

***From (3.3)***

$$0 \le \chi^2(p,q) \le 2\chi^2(p,q)$$

$$0 \le \chi^2(p,q)$$

**Example: 3**considering a convex function

$$\Phi(t) = (t-1)^2, \quad \forall\ t > 1$$

$$\Phi'(t) = 2(t-1), \quad \Phi(1) = 0 = \Phi'(1)$$

$$\Phi''(t) = 2 > 0$$

Then we can find out the Csiszar's $\Phi$ -divergence measure

$$I_\phi(p,q) = \sum_{i=1}^{n} \frac{p_i^2}{q_i} - 1 = \chi^2(p,q)$$

$$I_{\phi'}(p,q) = 2\left[ \sum_{i=1}^{n} q_i \left( \frac{p_i}{q_i} - 1 \right) \right]$$

$$I_{\phi'}(p,q) = 2\left( \sum_{i=1}^{n} p_i - \sum_{i=1}^{n} q_i \right)$$

$$I_{\phi'}(p,q) = 0$$

$$I_{\phi'}(\frac{p^2}{q}, p) = 2\sum_{i=1}^{n} p_i \left( \frac{p_i^2}{q_i} . \frac{1}{p_i} - 1 \right) = 2\sum_{i=1}^{n} \frac{p_i^2}{q_i} - 2$$

***From (3.1)***

$$I_{\phi'}(\frac{p^2}{q}, p) = 2\left( \sum_{i=1}^{n} \frac{p_i^2}{q_i} - 1 \right)$$

$$I_{\phi'}(\frac{p^2}{q}, p) = 2\chi^2(P,Q)$$

$$\phi'(1)(P_n - Q_n) \le I_\phi(P,Q) - Q_n\phi(1)$$

$$\le I_{\phi'}(\frac{p^2}{q}, p) - I_{\phi'}(P,Q)$$

$$0 \le \chi^2(P,Q) - 0 \le 2\chi^2(P,Q) - 0$$

$$0 \le \chi^2(P,Q) \le 2\chi^2(P,Q)$$

$$0 \le \chi^2(P,Q)$$

***From (3.2)***

$$0 \le I_{\phi}(P,Q) - Q_n \Phi(\frac{P_n}{Q_n})$$

$$\le I_{\phi^{'}}(\frac{p^2}{q}, p) - \frac{P_n}{Q_n} I_{\phi^{'}}(P,Q)$$

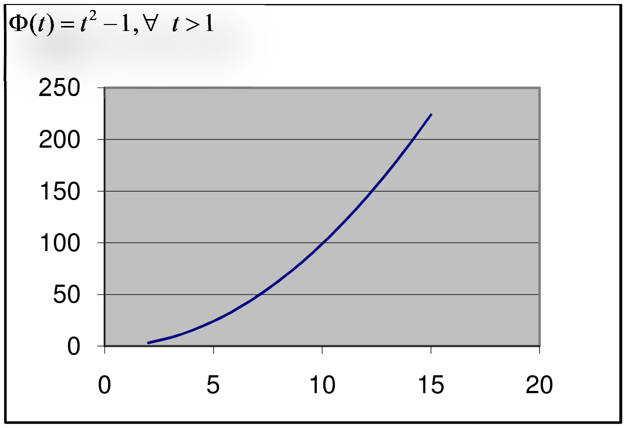


**Figure 3.1**

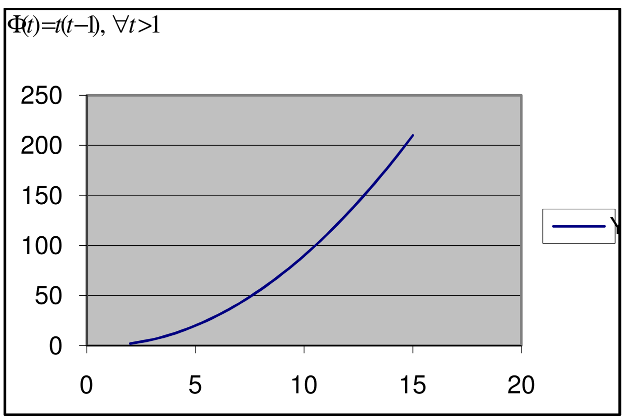


**Figure 3.3**

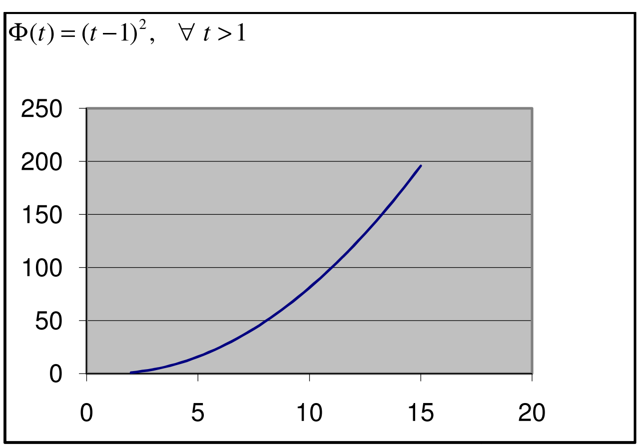


**Figure 3.3**

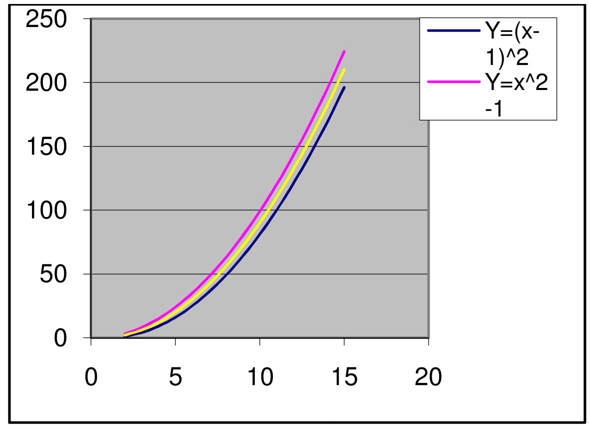


**Figure 3.4: Combined graph of following functions**

## 4. BOUNDS OF $\chi^2$ -DIVERGENCE MEASURE:

**Proposition: 4.1**Let $P,Q \in \Gamma_n$ be two probability distribution then we have the following relations

$$\chi^2(P,Q) > K(P,Q) \quad (4.1)$$

**Proof:-**Considering a inequality

$$t^2 - 1 > t \log t, \quad \forall t > 1$$

Putting $t = \frac{p_i}{q_i}$ and applying Csiszar's $\Phi$ -divergence measure in above inequality

$$\frac{p_i^2}{q_i^2} - 1 > \frac{p_i}{q_i} \log \frac{p_i}{q_i}$$

$$\sum_{i=1}^{n} q_i \left( \frac{p_i^2}{q_i^2} - 1 \right) > \sum_{i=1}^{n} q_i \left( \frac{p_i}{q_i} \log \frac{p_i}{q_i} \right)$$

$$\sum_{i=1}^{n} \frac{p_i^2}{q_i} - 1 > \sum_{i=1}^{n} p_i \log \frac{p_i}{q_i}$$

$$\chi^2(P,Q) > K(P,Q)$$

where '$K(P,Q)$' is Kullback and Leibler's divergence measure

**Proposition: 4.2** Let $P,Q \in \Gamma_n$ be two probability distribution then we have the following relation

$\chi^2(P,Q) > D(P,Q)$ (4.2)

**Proof:-**Considering an inequality

$$t^2 - 1 > (t-1)\log\left(\frac{1+t}{2}\right), \ \forall t > 1$$

Putting $t = \frac{p_i}{q_i}$ and applying Csiszar's $f$ -divergence measure in above inequality

$$\frac{p_i^2}{q_i^2} - 1 > \left(\frac{p_i}{q_i} - 1\right)\log\left(\frac{\frac{p_i}{q_i}+1}{2}\right)$$

$$\sum_{i=1}^{n} q_i\left(\frac{p_i^2}{q_i^2} - 1\right) > \sum_{i=1}^{n} q_i\left(\frac{p_i - q_i}{q_i}\right)\log\left(\frac{p_i + q_i}{2q_i}\right)$$

$$\sum_{i=1}^{n}\frac{p_i^2}{q_i} - 1 > \sum_{i=1}^{n}(p_i - q_i)\log\left(\frac{p_i + q_i}{2q_i}\right)$$

$$\chi^2(P,Q) > D(P,Q)$$

Where '$D(P,Q)$' is Relative J-divergence measure

**Proposition: 4.3** Let $P, Q \in \Gamma_n$ be two probability distribution then we have the following relation

$\chi^2(P,Q) > W(P,Q)$ (4.3)

**Proof: -**Considering a inequality

$$t^2 - 1 > \frac{2t}{1+t}, \ \forall t > 1$$

Putting $t = \frac{p_i}{q_i}$ and using Csiszar's $\Phi$ -divergence properties in above inequality

$$\frac{p_i^2}{q_i^2} - 1 > \frac{2\frac{p_i}{q_i}}{1+\frac{p_i}{q_i}}$$

$$\sum_{i=1}^{n} q_i\left(\frac{p_i^2}{q_i^2} - 1\right) > \sum_{i=1}^{n} q_i\left(\frac{2p_i}{p_i + q_i}\right)$$

$$\sum_{i=1}^{n} q_i\left(\frac{p_i^2}{q_i^2} - 1\right) > \sum_{i=1}^{n}\frac{2p_i q_i}{p_i + q_i}$$

$$\sum_{i=1}^{n}\frac{p_i^2}{q_i} - 1 > \sum_{i=1}^{n}\frac{2p_i q_i}{p_i + q_i}$$

$$\chi^2(P,Q) > W(P,Q)$$

Where '$W(P,Q)$, is well known Harmonic mean divergence measure

**Proposition: 4.4** Let $P, Q \in \Gamma_n$ be two probability distribution then we have the following relation

$\chi^2(P,Q) > F(P,Q)$ (4.4)

**Proof:**Considering a inequality

$$t^2 - 1 > t\log\frac{2t}{1+t}, \ \forall t > 1$$

Putting $t = \frac{p_i}{q_i}$ and applying Csiszar's $\Phi$ -divergence measure in above inequality

$$\frac{p_i^2}{q_i^2} - 1 > \frac{p_i}{q_i}\log\frac{2\frac{p_i}{q_i}}{1+\frac{p_i}{q_i}}$$

$$\sum_{i=1}^{n} q_i\left(\frac{p_i^2}{q_i^2} - 1\right) > \sum_{i=1}^{n} q_i\left(\frac{p_i}{q_i}\log\frac{2p_i}{p_i + q_i}\right)$$

$$\sum_{i=1}^{n} q_i\left(\frac{p_i^2}{q_i^2} - 1\right) > \sum_{i=1}^{n} p_i \log\frac{2p_i}{p_i + q_i}$$

$$\sum_{i=1}^{n}\frac{p_i^2}{q_i} - 1 > \sum_{i=1}^{n} p_i \log\frac{2p_i}{p_i + q_i}$$

$$\chi^2(P,Q) > F(P,Q)$$

Where '$F(P,Q)$' is the Relative Jenson-Shannon divergence measure.

**Proposition: 4.5** Let $P, Q \in \Gamma_n$ be two probability distribution then we have the following relation

$\chi^2(P,Q) > J(P,Q)$ (4.5)

**Proof:**Considering a inequality

$$t^2 - 1 > (t-1)\log t, \ \forall t > 1$$

Putting $t = \frac{p_i}{q_i}$ and applying Csiszar's $f$ -divergence measure

$\frac{p_i^2}{q_i^2} - 1 > \left(\frac{p_i}{q_i} - 1\right) \log \frac{p_i}{q_i}$ in above inequality

$$\sum_{i=1}^{n} q_i \left(\frac{p_i^2}{q_i^2} - 1\right) > \sum_{i=1}^{n} q_i \left(\frac{p_i - q_i}{q_i}\right) \log \frac{p_i}{q_i}$$

$$\sum_{i=1}^{n} \frac{p_i^2}{q_i} - 1 > \sum_{i=1}^{n} (p_i - q_i) \log \frac{p_i}{q_i}$$

$$\chi^2(P,Q) > J(P,Q)$$

Where '$J(P,Q)$' is the Jenson divergence measure

**Proposition: 4.6** Let $P, Q \in \Gamma_n$ be two probability distribution then we have the following relation

$\chi^2(P,Q) > B(P,Q)$ (4.6)

**Proof:**Considering a inequality

$$t^2 - 1 > \sqrt{t}, \ \forall t > 1$$

Putting $t = \frac{p_i}{q_i}$ and applying Csiszar's $\Phi$ -divergence measure in above inequality

$$\frac{p_i^2}{q_i^2} - 1 > \sqrt{\frac{p_i}{q_i}}$$

$$\sum_{i=1}^{n} q_i \left(\frac{p_i^2}{q_i^2} - 1\right) > \sum_{i=1}^{n} q_i \sqrt{\frac{p_i}{q_i}}$$

$$\sum_{i=1}^{n} \frac{p_i^2}{q_i} - 1 > \sum_{i=1}^{n} \sqrt{p_i q_i}$$

$$\chi^2(P,Q) > B(P,Q)$$

Where '$B(P,Q)$' be the Bhattacharya Divergence measure

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