Fritz John Type Optimality Criteria for Optimal Solution of Nonlinear Programming Problem

Prashant Chauhan[1], Paras Bhatnagar[2]

[1]*Deptt. of Mathematics, IFTM University, Lodhipur Rajput, Moradabad*
[2]*Deptt. of Mathematics, GLBITM, Greater Noida*

Abstract: **This paper is proposed to study Fritz-John optimality and duality results for nonlinear programming problem. Weir and Mond provided a Fritz John dual for the nonlinear programming problems involving differentiable functions by using the Fritz John optimality conditions instead of Kuhn-Tucker conditions and thus did not require a constraint qualification. Here we associate the Mond Wier type Fritz John dual to the nonlinear programming problem.**

1. INTRODUCTION

Weir and Mond [7] provided a Fritz John dual for the nonlinear programming problems involving differentiable functions by using the Fritz John [4] optimality conditions instead of Kuhn-Tucker [5] conditions and thus did not require a constraint qualification.

Keywords: Nonlinear programming, Convex functions, Fritz-John optimality and duality conditions, Kuhn-Tucker conditions.

1. Introduction

We obtain Fritz John type necessary optimality criteria for the optimal solution of the following nonlinear program:

(P) Min $f(x)$

Subject to $g_j \le 0$, $j = 1,2,3.......,m$, $x \in S$,

Where $S \subseteq R^n$, $f : S \to R$, and $g_j : S \to R$ $j = 1,2,3.......,m$ are real valued functions. S is a locally connected set such that for each $x^*, x \in S$, there exists a vector value function $H_{x^*,x^{(\lambda)}}$, satisfying:

$$H_{x^*,x^{(\lambda)}} \in S, 0 < \lambda < a \qquad (1.1)$$

Let $X^0 = \{x \in S / g_j \le 0, j = 1,2,3.......,m\}$ $H_{x^*,x}$ is continuous in the interval $]0, a(x^*,x)[$ and

$$H_{x^*,x^{(0)}} = x^*,\ H_{x^*,x^{(1)}} = x \qquad (1.2)$$

And the right differentials of f and g_j, $j = 1,2,3.......,m$ at x^* exist with respect to the arc $H_{x^*,x^{(\lambda)}}$.

Theorem-1 Let x^*be an optimal solution of (P). If $(df)^+(x^*, H_{x^*,x^{(0+)}})$ and $(dg_I)^+(x^*, H_{x^*,x^{(0+)}})$ are convex functions of x and g_j, $j \in J$ is continuous at $x^*$with S convex or $S = R^n$ then there exist $r_0^* \in R$, $r^* \in R$, Such that

$$r_0^*(df)^+(x^*, H_{x^*,x^{(0+)}}) + (r_0^*)^T (dg_I)^+(x^*, H_{x^*,x^{(0+)}}) \ge 0, \text{ for all } x \in S \qquad (1.3)$$

$$(r_0^*)^T g(x^*) = 0 \qquad (1.4)$$

$$(r_0^*, r^*) \ge 0 \qquad (1.5)$$

Where $I = I(x^*) = \{i / g_i(x^*) = 0\}$

and $J = J(x^*) = \{j / g_j(x^*) < 0\}$.

Proof: First we shall show that the system

$$\left\{\begin{array}{l}(df)^+(x^*, H_{x^*,x^{(0+)}}) < 0 \\ (dg_I)^+(x^*, H_{x^*,x^{(0+)}}) < 0\end{array}\right\} \qquad (1.6)$$

Has no solution in $x \in S$.

If possible let $x \in S$ be a solution of the system (1.6). Since right differentials of f and g_j, $i \in I$ at x^* exist with respect to the arc $H_{x^*,x^{(\lambda)}}$. Therefore

$$f(H_{x^*,x^{(\lambda)}}) = f(x^*) + \lambda (df)^+(x^*, H_{x^*,x^{(0+)}}) + \lambda \alpha(\lambda) \qquad (1.7)$$

and $g_i(H_{x^*,x^{(\lambda)}}) = g_i(x^*) + \lambda(dg_i)^+(x^*, H_{x^*,x^{(0+)}})$
$+\lambda\alpha_i(\lambda)$ (1.8)

where $\alpha:[0,1] \to R$, $\lim_{\lambda\to 0} \alpha(\lambda) = 0$ (1.9)

$\alpha_i:[0,1] \to R$, $\lim_{\lambda\to 0^+} \alpha_i(\lambda) = 0$, $i \in I(x^*)$ (1.10)

Using (1.6),(1.9) and (1.10) we get for small enough λ, say $0 < \lambda < \lambda_0$ then $(df)^+(x^*, H_{x^*,x^{(0+)}}) + \alpha(\lambda) < 0$ and

$(dg_i)^+(x^*, H_{x^*,x^{(0+)}}) + \alpha_i(\lambda) < 0$, $i \in I(x^*)$

hence it follows by using the relation(1.7) and (1.8) that for $0 < \lambda < \lambda_0$,

$f(H_{x^*,x^{(\lambda)}}) - f(x^*) < 0$ (1.11)

$g_i(H_{x^*,x^{(\lambda)}}) - g_i(x^*) < 0$, $i \in I(x^*)$ (1.12)

Now g_j , $j \in J$ is continuous at x^* and $H_{x^*,x^{(\lambda)}}$ is also a continuous function of λ. Therefore

$\lim_{\lambda\to 0^+} g_j(H_{x^*,x^{(\lambda)}}) = g_j(x^*) < 0$

Which implies that there exist λ_j^*,

$0 < \lambda_j^* < a(x^*, x)$, $j \in J$ such that

$g_j(H_{x^*,x^{(\lambda)}}) < 0$ for $0 < \lambda < \lambda_j^*$ (1.13)

Let $\lambda^* = \min(\lambda_0, \lambda_j^*, j \in J)$ then from (1.11) to (1.13) it follows that for $0 < \lambda < \lambda_j^*$, $H_{x^*,x^{(\lambda)}} \in X^0$ and $f(H_{x^*,x^{(\lambda)}}) < f(x^*)$, which is a contradiction as x^* is an optimal solution of (P).

Hence the system (1.6) has no solution $x \in S$.

Since $(df)^+(x^*, H_{x^*,x^{(0+)}})$ and $(dg_I)^+(x^*, H_{x^*,x^{(0+)}})$, $i \in I(x^*)$ are convex functions of x therefore there exist $r_0^* \in R$, $r_i^* \in R$, $i \in I$ such that

$r_0^*(df)^+(x^*, H_{x^*,x^{(0+)}}) + r_I^{*T}(dg_I)^+(x^*, H_{x^*,x^{(0+)}}) \geq 0$ for all $x \in S$,where $(r_0^*, r_I^*) \geq 0$.

Defining $r_j^* = 0$ we get the required result.

Now we associate the following Mond Wier type Fritz John dual to the problem (P):

(D) Maximize $f(\mu)$

Subject to

$r_0(df)^+(\mu, H_\mu, x^{(0+)}) + r^T(dg)^+(\mu, H_\mu, x^{(0+)}) \geq 0$ for all $x \in X^0$ (1.14)

$\sum_{j=1}^{m} r_j g_j(\mu) \geq 0$ (1.15)

$\mu \in S$, $(r_0, r) \geq 0$, $r_0 \in R$, $r \in R^m$ (1.16)

Theorem-2 (Weak duality) Let x be feasible for (P) and (μ, r_0, r) be feasible for (D). If f is locally P-connected and $\sum_{j=1}^{m} r_j g_j(\mu)$ is strongly P-connected at μ then $f(x) \geq f(\mu)$.

Proof: If possible let $f(x) < f(\mu)$. Since f is locally P-connected at μ therefore it follows that

$r_0(df)^+(\mu, H_{\mu,x^{(0+)}}) \leq 0$ (1.17)

With strict inequality if $r_0 > 0$ .by the feasibility of x and (μ, r_0, r) for (P) and (D) respectively we get

$$\sum_{j=1}^{m} r_j g_j(x) \leq \sum_{j=1}^{m} r_j g_j(\mu),$$

Now $\sum_{j=1}^{m} r_j g_j$ is strongly locally P-connected at μ so,

$d(\sum_{j=1}^{m} r_j g_j)^+(\mu, H_{\mu,x^{(0+)}}) \leq 0$ (1.18)

With strict inequality if some $r_j > 0$, $j = 1,2,.....,m$. Adding (1.17) ,(1.18) and using (1.16) we get

$r_0(df)^+(\mu, H_{\mu,x^{(0+)}}) + r^T(dg)^+(\mu, H_{\mu,x^{(0+)}}) < 0$

Which is a contradiction to (1.14).

Hence $f(x) \geq f(\mu)$.

Theorem-3: (Strong Duality) Let x^* be an optimal solution of (P), $(df)^+(x^*, H_{x^*,x^{(0+)}})$ and $(dg_I)^+(x^*, H_{x^*,x^{(0+)}})$ be the convex functions of x and g_j $j \in J$ be continuous at x^* with S convex or $S = R^n$. Then there exist $r_0^* \in R$ and $r_0^* \in R^m$ such that (x^*, r_0, r^*) is feasible for (D) and the values of the objective functions of (P) and (D) are at x^*. Also if for each feasible (μ, r_0, r) for (D), f is locally P-connected and $\sum_{j=1}^{m} r_j g_j$ is strongly locally P-connected at μ then (x^*, r_0^*, r^*) is optimal for (D).

Proof: Since x^* is an optimal solution of (P) therefore by theorem -1, there exist $r_0^* \in R$ and $r_0^* \in R^m$ such that (x^*, r_0^*, r^*) is feasible for (D). Equality of objective functions for (P) and (D) follows trivially. Further if (x^*, r_0^*, r^*) is not optimal for (D) then there exists (μ, r_0, r) ,feasible for (D) such that $f(\mu) > f(x^*)$.which is a contradiction to weak duality.

2. CONCLUSION

In theorem-1, we obtained Fritz John type necessary optimality condition for nonlinear programming problem (P). In theorem-2 and 3, we obtained Mond Wier type duality conditions for nonlinear program by considering a dual (D) of (P).

REFERENCE

[1] Bazaraa, M. S. and Shetty, C.M., Nonlinear Programming, Theory and algorithms, John Wiley and Sons 1979.

[2] Dantzig, G. B., Eisenberg, E. and Cottle, R. W., Symmetric dual of Nonlinear Programs, Pacific J. Math.15, 809-812,(1965).

[3] Faigle, U., Kern K., Still, G., Algorithm Principles to Mathematical Programming, Kluwer Academic publishers, Dordrecht,2002.

[4] John, F., “Extremum Problems with Inequalities as Subsidiary Conditions,” in Studies and Essays: Courant Anniversary Volume, K. O. Friedrichs, Neugebauer, O. E., and Stoker, J. J., (Eds.), Wiley-Interscience, N. Y., pp. 187-204, 1948.

[5] Kuhn H. W. and Tucker, A. W., “Nonlinear Programming,” Proceedings of second Berkeley Symposium on Mathematical Statistics and Probability, (University of California Press 1951), 481-492

[6] Mond B. and Egudo, R. R., “On strict converse duality in nonlinear programming,” J. Inform. Optim. Sci.6 (1985), 113-16.

[7] Mond B. and Weir, T., “Generalized concavity and duality,” in Generalized concavity in optimization and economics (eds. S. Schaible and W.T. Ziemba), (Academic Press 1981), 263-279.