On Petri Net Generating all the Standard Basis for n-Dimensional Euclidean Space

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Abstract: **Petri nets are the prominent tools of current re- search in branch of discrete mathematics. They are invented by Carl Adam Petri in 1962, a tool for the study of certain discrete dynamical systems. Petri nets form directed bipartite graphs, together with an initial marking. Petri nets have a well known graphical representation in which transitions are represented as boxes and places as circles with directed arcs interconnecting them, to represent the flow relation between them. Petri net can be used as graphical tool to planning and designing such a system with given objectives, more effectively than flowcharts and block diagrams. In this paper, we will discuss a special class of Petri net, called 1-safe Petri net that contains '0', '1' digits as marking vectors in its reachability tree. Here, we will prove that a disconnected 1-safe Petri nets whose reachability tree on the firing of transitions at the $(n-1)_{th}$ stage contain all the standard basis e_i, $1 \leq i \leq n$ for n-dimensional Euclidean Space.**

Keywords: **1-safe Petri nets, directed graph, binary vectors, basis, Euclidean space**

1. INTRODUCTION

Petri nets are a graphical and mathematical tool applicable to model various kinds of dynamic event-driven systems such as computer networks etc. They are invented by Carl Adam Petri [2] in 1962 for describing and studying information processing systems that are characterized as concurrency, parallelism, synchronization etc. As a graphical tool, Petri net can be used for planning and designing a system with given objectives more practically effective than flowchart and block design diagrams. As a mathematical tool, it enables one to setup state equations, algebraic equations and other mathematical models which are govern the behavior of discrete dynamical systems. Petri nets also are useful for both practitioners as well as theoreticians. Practitioners can learn from theoreticians how to make their models more methodical, and theoreticians can learn from practitioners how to make their model more realistic. In this paper, the existence of a disconnected 1-safe Petri net which is isomorphic to n copies of graph $K2$ has been shown and its reachability tree on the firing of transitions at the $(n-1)th$ stage contains all the standard basis ei, $1 \leq i \leq n$ for n-dimensional Euclidean Space. In n-dimensional Euclidean space, the standard basis consists of n-distinct vectors $\{ei : 1 \leq i \leq n\}$, where ei denotes the vector with 1 in the ith coordinate and $0's$ elsewhere. These vectors are a basis in the sense that any other vectors can be expressed uniquely as a linear combination of these.

2. PRELIMINARIES

For standard terminology and notation on Petri nets, we refer the reader to Peterson [3]. In this paper, for the sake of better understanding of proof, we shall adopt the Jenson's matrix[1] form definition, which is as follows:

A *Petri net* is a 5-tuple $C = (P; T; I-; I+; \_0)$, where

(a) P is a nonempty set of 'places',
(b) T is a nonempty set of 'transitions',
(c) $P \cap T = \emptyset$,
(d) $I-; I+ : P \times T \longrightarrow N$, where N is the set of nonnegative integers, are called the *negative* and the *positive* 'incidence functions' (or, 'flow functions') respectively,
(e) $\forall p \in P; \exists t \in T : I-(p; t) \neq 0$ or $I+(p; t) \neq 0$ and
$\forall t \in T; \exists p \in P : I-(p; t) \neq 0$ or $I+(p; t) \neq 0$,
(f) $\_0 : P \rightarrow N$ is the *initial marking*.

In fact, $I-(p; t)$ and $I+(p; t)$ represent the number of arcs from p to t and t to p respectively. $I-$, $I+$ and $\_0$ can be viewed as matrices of size $|P| \times |T|$, $|P| \times |T|$ and $|P| \times 1$, respectively.

As in many standard books (e.g., see [4]), Petri net is a particular kind of directed graph, together with an initial marking $\_0$. The underlying graph of a Petri net is a directed, weighted, bipartite graph consisting of two kinds of nodes, called places and transitions, where arcs are either from a place to a transition or from a transition to a place. Hence, Petri nets have a well known graphical representation in which transitions are represented as boxes and places as circles with directed arcs interconnecting places and transitions to represent the flow relation. The initial marking is represented by placing a token in the circle representing a place pi as a black dot whenever $\mu^{O}(pi) = 1$, $1 \leq i \leq n = |P|$. In general, a *marking* _ is a mapping $\mu: P \rightarrow N$. A marking μ can hence be represented as a vector $\mu \in Nn$; $n = |P|$, such that the *ith* component of $\mu$is the value $\mu(pi)$.

Let $C = (P; T; I-; I+; \mu)$ be a Petri net. A transition $t \in T$ is said to be *enabled* at μif and only if $I-(p; t) \leq \mu(p)$, $\forall p \in P$. An enabled transition may or may not 'fire' (depending on whether or not the event actually takes place). After firing at μ, the new marking μ' is given by the rule

We say that *t fires at* $\mu$to yield μ' (or, that *t fires* $\mu$to μ'), and we write $\mu \longrightarrow t\, \mu'$, whence μ' is said to be *directly reachable* from μ. Hence, it is clear, what is meant by a sequence like

$$\mu^0 \xrightarrow{t_1} \mu^1 \xrightarrow{t_2} \mu^2 \xrightarrow{t_3} \mu^3 \ldots \xrightarrow{t_k} \mu^k,$$

which simply represents the fact that the transitions $t_1, t_2, t_3, \ldots, t_k$ have been successively fired to transform the marking μ^0 into the marking μ^k. The whole of this sequence of transformations is also written in short as $\mu^0 \xrightarrow{\sigma} \mu^k$, where $\sigma = t_1, t_2, t_3, \cdots, t_k$.

A marking μ is said to be *reachable from* μ^0, if there exists a sequence of transitions which can be successively fired to obtain μ from μ^0. The set of all markings of a Petri net C reachable from a given marking μ is denoted by $M(C, \mu)$ and, together with the arcs of the form $\mu^i \xrightarrow{t_r} \mu^j$, represents what in standard terminology called the *reachability graph* $R(C, \mu)$ of the Petri net C. If the reachability graph has no cycle then it is called *reachability tree* of the Petri net C.

A place in a Petri net is *safe* if the number of tokens in that place never exceeds one. A Petri net is *safe* if all its places are safe.

Let $C = (P, T, I^-, I^+, \mu^0)$ be a Petri net with $|P| = n$ and $|T| = m$, the incidence matrix $I = [a_{ij}]$ is an $n \times m$ matrix of integers, $|P| = n$ and $|T| = m$ and its entries are given by $a_{ij} = a_{ij}^+ - a_{ij}^-$ where $a_{ij}^+ = I^+(p_i, t_j)$ is the number of arcs from transition t_j to its output place p_i, known as positive incidence matrix and $a_{ij}^- = I^-(p_i, t_j)$ is the number of arcs from place p_i to its output transition t_j, known as negative incidence matrix. In other words, $I = I^+ - I^-$

The *preset* of a transition t is the set of all input places to t, i.e., ${}^\bullet t = \{p \in P : I^-(p, t) > 0\}$. The *postset* of t is the set of all output places from t, i.e., $t^\bullet = \{p \in P : I^+(p, t) > 0\}$. Similarly, p'*s* preset and postset are ${}^\bullet p = \{t \in T : I^+(p, t) > 0\}$ and $p^\bullet = \{t \in T : I^-(p, t) > 0\}$, respectively.

The *Hamming distance* between two bit-strings $u = u_1, u_2, \cdots, u_n$, $v = v_1, v_2, \cdots, v_n \in \{0, 1\}^n$ of length n is the number of bit positions in which u and v differ: $d_H^n(u, v) = |\{i \in \{1, 2, \cdots, n\} : u_i \neq v_i\}|$. The Hamming distance between 1011101 and 1001001 is two.

Let x_1, x_2, x_3 $\cdots$, x_n be vectors from n-dimensional Euclidean space and let α_1, α_2, α_3, $\cdots$, α_n be a system of scalars (real numbers). Then the vector

$$X = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \cdots + \alpha_n x_n$$

is called a *linear combination* of the vectors x_1, x_2, x_3 $\cdots$, x_n. We also say that X is generated by the vectors x_1, x_2, x_3 $\cdots$, x_n.

If the linear combination $\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \cdots + \alpha_n x_n = 0$ implies that all the $\alpha_i' s = 0$, $\forall n$ then the linear combination is called *linearly independence*.

Let $B = \{x_1, x_2, x_3, \cdots, x_n\}$ be a finite subset of n-dimensional Euclidean space. Then B is called a *basis* if B spans Euclidean space and B is linearly independent.

3. SOME BASIC RESULTS

In this section, we discuss some observational results for 1-safe Petri net whose reachability tree contains standard basis that are the main results for this paper.

Theorem 3.1: Let $C = (P, T, I^-, I^+, \mu^0)$ with $|P| = n$ be a disconnected 1-safe Petri net isomorphic to nK_2 :⊙⟶□. Then firing of the transitions at the initial marking $\mu^0(p) = 1$, $\forall p \in P$ in the $(n-1)^{th}$ stage generate every standard basis e_i, $1 \leq i \leq n = |P|$, for n-dimensional Euclidean space.

Proof: Let $C = (P, T, I^-, I^+, \mu^0)$ be a disconnected 1-safe Petri net isomorphic to nK_2 :⊙⟶□.. Since $\mu^0(p) = 1$ $\forall p \in P$, all the transitions are enabled. In the first step of firing of n transitions, we get ${}^nC_1$ distinct binary n-vectors whose Hamming distance is 1 from the initial marking vector. At each of these n marking vectors, $(n-1)$ transitions are enabled, and after firing give at least ${}^nC_2$ distinct marking vectors each of whose Hamming distance is 2 from the initial marking.

In general at stage $(n-1)$, we get a set of at least ${}^nC_{n-1}$ new distinct binary n-vectors whose Hamming distance is $n-1$ from the initial marking, i.e., these have 1 in the i^{th} position otherwise 0, Hence, they are the required standard basis e_i, $1 \leq i \leq n = |P|$, for n-dimensional Euclidean space.

Theorem 3.2: Let $C = (P, T, I^-, I^+, \mu^0)$, $|P| = n$ be a 1-safe Petri net with $\mu^0(p) = 1$, $\forall p \in P$, then all the standard basis as the marking vectors in its reachability tree can be generated in either first stage of firing or $(n-1)^{th}$ stage of firing.

Proof: Let $C = (P, T, I^-, I^+, \mu^0)$, $|P| = n$ be a 1-safe Petri net with $\mu^0(p) = 1$, $\forall p \in P$. If $|T| < |P|$ then all the standard basis cannot be generated. Suppose $|T| \geq |P|$ then there will be two cases. In case one, all the standard basis will be generate in to the first step of firing of the transitions and in other case, by remark 3.4.2 in [5], which is stated as: In any 1-safe Petri net $C = (P, T, I^-, I^+, \mu^0)$, $|P| = |T| = n$, with $\mu^0(p) = 1$, $\forall p \in P$, at any stage in the dynamics of C when binary n-vectors at Hamming distance k from μ^0 are being generated all the binary n-vectors of Hamming distance less than k have already been generated. In this way at the $(n-1)^{th}$ stage we get the binary vectors whose Hamming distance from the initial marking is $(n-1)$ which are the required standard basis e_i, $1 \leq i \leq n = |P|$, for n-dimensional Euclidean space. ■

Proposition 3.1: If $I^-(p_i, t_j)$=1 $\forall i = j$ otherwise 0 $\forall i \neq j$ and $I^+(p_i, t_j)$=0, $\forall i, j$, then the Petri net C obtained by the incidence matrix $I = I^+ - I^-$ at the $\mu^0(p) = 1$,$\forall p \in P$ in the $(n-1)^{th}$ stage of firing of transitions contain the same number of standard basis as its incidence matrix.

Proof: Since $I^-(p_i, t_j)$=1 $\forall i = j$ otherwise 0 $\forall i \neq j$ and $I^+(p_i, t_j)$=0, $\forall i, j$ imply that $I = I^+ - I^- = -I^-$ which is corresponds to the identity matrix I_n. We know that an identity matrix I_n has n linearly independence columns. If each of the columns consider as the vectors e_i $\forall i$. Further, the Petri net C obtained by the incidence matrix is isomorphic to nK_2 :$\odot \longrightarrow \square$. By Theorem 3.1, we get n standard basis e_i, $1 \leq i \leq n = |P|$, for n-dimensional Euclidean space. Therefore, C has the same number of standard basis as its incidence matrix.

Theorem 3.3: Any Petri net C obtained by $|^\bullet t| = |t^\bullet| = 1$ and $|^\bullet p| = |p^\bullet| = 1$ then at the initial marking $\mu^0(p) = e_i$, i.e., i^{th} component is 1 otherwise 0. Then the reachability tree contains every standard basis e_i, $1 \leq i \leq n = |P|$, for n-dimensional Euclidean space.

Proof: Since $|^\bullet t| = |t^\bullet| = 1$ and $|^\bullet p| = |p^\bullet| = 1$. Then it will form a cyclic in the Petri net. Further, $\mu^0(p) = e_i$ then the transition t_i is enabled and fire. After firing, we will get $\mu^1(p) = e_{i+1}$, and continuing this process till all the standard basis e_i, $1 \leq i \leq n = |P|$ obtained. ■

4. CONCLUSIONS

In this paper, we have discussed the generation of the standard basis for n-dimensional Euclidean space using 1-safe Petri nets. As we know that these standard basis are very much useful in vector space for the linear transformations of vectors.

One think about the characterize those Petri nets which has all the standard basis as the marking vectors. Further, we can also think about which set of vectors are linearly dependence and independence.

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