On the $\boldsymbol{k}$-Numbers of Some Generalized Kuratowski Operators in Topology

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Abstract: **If $X$is a topological space and $A \subseteq X$, then the number of distinct setsthat can be obtained from A by using all possible compositions of pre-closure and complement is at most 10. Similarly, this number for $c_\beta, c_\alpha, and\ c_\sigma$is 8,14 and 10 respectively. Explicit expressions for these sets areprovided. An example is also provided where all these different sets are realized.A collection of all the semi groups (monoids) of the monoid generated by c_α, that is$\mathcal{M}_\alpha$ is provided.**

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1. INTRODUCTION

The Kuratowski closure-complement Theorem [7], a result of basic point set topology, was first proposed and proved by Kazimierz Kuratowski**.** Since then a lot of research has been carried out on Kuratowski closure operators within and outside the realm of general topology [2,4,12].

In this paper, we investigate the same for some common generalized closure operators, namely, π-, σ-, α-, β-closures. It is found that the maximum number of distinct sets that can be obtained by repeatedly taking closure and complement on a set is 10, 10, 14 and 8 respectively in case of π-closure, σ-closure, α-closure, and β-closure respectively. It is also found that in all these four cases, the generalized Kuratowski operators obtained in the process give rise to a monoids under compositions relation. The generators of these monoids have been obtained and some semi groups contained in these monoids are studied in the paper.

We recall some known definitions:

Definition 1.1 Let (X, τ) be a topological space. Then a subset A of (X, τ) is called

i.)semi-open[8] if $A \subseteq cl\ int(A)$;

ii.)α-open [10] if $A \subseteq int\ cl\ int(A)$;

iii.)pre-open[9] if $A \subseteq int\ cl(A)$;

iv.)β-open[1] if $A \subseteq cl\ int\ cl(A)$.

The complement of a semi-open (resp. α-open, pre-open, β-open) set is known as semi-closed (resp. α-closed, pre-closed, β-closed) set.

Intersection of all the semi-closed (resp. pre-closed, α-closed and β-closed) sets containingthe set A is called the semi-closure (resp. pre-closure, α-closure and β-closure) of A and denoted by $c_\sigma(A)\left(\text{resp.}\ c_\pi(A), c_\alpha(A)\ \text{and} c_\beta(A)\right)$.

Theorem 1.2[3]*In a topological space (X, τ) with $A \subseteq X$,we have*

i.) $c_\pi(A) = A \cup cl\ int\ (A)$;

ii.) $c_\beta(A) = A \cup int\ cl\ int\ (A)$;

iii.) $c_\alpha(A) = A \cup cl\ int\ cl\ (A)$;

iv.) $c_\sigma(A) = A \cup int\ cl\ (A)$;

2. $\boldsymbol{k}$-NUMBERS OF GENERALIZED CLOSURE OPERATORS

In this section, first we define the$k$-numberin a topology.

Definition 2.1 Let(X, τ)be a topological space. A set$A \subseteq X$and c_j be a generalized closure operator on (X, τ) . Then the maximum number of distinct sets that can be generated from A by successive applications of c_j and the complement operator$c$ is called the$k$*-number* of c_j and is denoted by $k(c_j)$.

Theorem 2.2*Let c_j be a generalized closure operator in (X, τ). Then we have $k(c_j) = 10{,}8{,}14\ and\ 10$ for $j = \pi, \beta, \alpha\ and\ \sigma$.*

Proof. *i.)***The case of c_π :**

For $A \subseteq X$, let us use the following notations:

$\pi_0(A) = A$ (the identity)	$\pi_4(A) = c.c_\pi(A)$
$\pi_1(A) = c(A)$ (complement)	$\pi_5(A) = c.c_\pi.c(A)$(pre-interior)
$\pi_2(A) = c_\pi(A)$(pre-closure)	$\pi_6(A) = c_\pi.c.c_\pi(A)$
$\pi_3(A) = c_\pi.c(A)$	$\pi_7(A) = c.c_\pi.c.c_\pi(A)$
$\pi_8(A) = c_\pi.c.c_\pi.c(A)$	$\pi_9(A) = c.c_\pi.c.c_\pi.c(A)$

Here "." denotes the composition operation. For example, $c_\pi.c(A)$denotes the pre-closure of the complement of$A$.

We have

$$\pi_2(A) = A \cup cl\ int(A)$$

$$\pi_3(A) = c_\pi . c(A) = c(A) \cup cl\ int\ \big(c(A)\big) = c(A) \cup cl. c\big(int(A)\big) = c(A) \cup c\big(int\ cl(A)\big) = c(A \cap int\ cl(A))$$

Procedding similarly, we have

$$\pi_4(A) = c\big(A \cup cl\ int\ (A)\big),\ \pi_5(A) = A \cap int\ cl(A)$$

$$\pi_6(A) = c\big([A \cup cl\ int(A)] \cap int\ cl(A)\big)$$

$$\pi_7(A) = [A \cup cl\ int(A)] \cap int\ cl(A)$$

$$\pi_8(A) = [A \cap\ int\ cl(A)] \cup\ cl\ int(A)$$

$$\pi_9(A) = c\big([A \cap\ int\ cl(A)] \cup\ cl\ int(A)\big)$$

Based on the above expressions for $\pi_i(A), i = 0,1, \dots\dots, 9$, we obtain the following composition table for the operators $\pi_0(A), \pi_1(A), \pi_2(A), \dots\dots\dots, \pi_9(A)$:

o	π_0	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9
π_0	π_0	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9
π_1	π_1	π_0	π_4	π_5	π_2	π_3	π_7	π_6	π_9	π_8
π_2	π_2	π_3	π_2	π_3	π_6	π_8	π_6	π_8	π_8	π_6
π_3	π_3	π_2	π_6	π_8	π_2	π_3	π_8	π_6	π_6	π_8
π_4	π_4	π_5	π_4	π_5	π_7	π_9	π_7	π_9	π_9	π_7
π_5	π_5	π_4	π_7	π_9	π_4	π_5	π_9	π_7	π_7	π_9
π_6	π_6	π_8	π_6	π_8	π_8	π_6	π_8	π_6	π_6	π_8
π_7	π_7	π_9	π_7	π_9	π_9	π_7	π_9	π_7	π_7	π_9
π_8	π_8	π_6	π_8	π_6	π_6	π_8	π_6	π_8	π_8	π_6
π_9	π_9	π_7	π_9	π_7	π_7	π_9	π_7	π_9	π_9	π_7

For the sake of convience, we just write π_i instead of $\pi_i(A)$. The readers may verify the above results for themselves. For example

$$\pi_4 \circ \pi_6(A) = c. c_\pi . c_\pi . c. c_\pi(A) = c. c_\pi . c. c_\pi(A) = \pi_7(A),$$

and so on.

From the above composition table, it is clear that $k(c_\pi) = 10$.

***ii.)The Case of c_β*:**

For $A \subseteq X$, we use the following notations:

$\beta_0(A) = A$ (the identity)

$\beta_1(A) = c(A)$ (the complement)

$\beta_2(A) = c_\beta(A)$ (β-closure)

$\beta_3(A) = c_\beta . c(A)$

$\beta_4(A) = c. c_\beta(A)$

$\beta_5(A) = c. c_\beta . c(A)$ (β-interior)

$\beta_6(A) = c_\beta . c. c_\beta(A)$

$\beta_7(A) = c. c_\beta . c. c_\beta(A)$

The set theoretic expressions for the above expressions are:

$$\beta_3(A) = c_\beta . c(A) = A \cup int\ cl\ int\ \big(c(A)\big) = A \cup int\ cl\ \big(c\big(cl(A)\big)\big) = A \cup int\ c\big(int\ cl(A)\big) = A \cup c\big(cl\ int\ cl(A)\big) = c(A \cap cl\ int\ cl\ (A))$$

Procceeding similarly, we have

$\beta_4(A) = c\big(A \cup int\ cl\ int(A)\big)$,

$\beta_5(A) = A \cap cl\ int\ cl(A)$,

$\beta_6(A) = c\big([A \cap cl\ int\ cl(A)] \cup int\ cl\ int\ (A)\big)$,

$\beta_7(A) = [A \cap cl\ int\ cl(A)] \cup int\ cl\ int\ (A)$.

The composition table of $\boldsymbol{\beta_i}$, $\boldsymbol{for\ i = 0, 1, 2 \dots\dots, 7}$ is the following:

∘	β_0	β_1	β_2	β_3	β_4	β_5	β_6	β_7
β_0	β_0	β_1	β_2	β_3	β_4	β_5	β_6	β_7
β_1	β_1	β_0	β_4	β_5	β_2	β_3	β_7	β_6
β_2	β_2	β_3	β_2	β_3	β_6	β_7	β_6	β_7
β_3	β_3	β_2	β_6	β_7	β_2	β_3	β_7	β_6
β_4	β_4	β_5	β_4	β_5	β_7	β_6	β_7	β_6
β_5	β_5	β_4	β_7	β_6	β_4	β_5	β_6	β_7
β_6	β_6	β_7	β_6	β_7	β_7	β_6	β_7	β_6
β_7	β_7	β_6	β_7	β_6	β_6	β_7	β_6	β_7

From the above table it follows that $k\big(c_\beta\big) = 8$.

***iii.) The case ofc_α and c_σ*:**

Proceeding as above, we can see that the number of distinct sets that can be obtained from c_α , c_σ and their complement are 14 and 10 respectively. For $A \subseteq X$, the notations for $c_\alpha$are used:

$\alpha_0(A) = A$ (the identity)	$\alpha_7(A) = c_\alpha . c. c_\alpha . c(A)$
$\alpha_1(A) = c(A)$ (complement)	$\alpha_8(A) = c. c_\alpha . c. c_\alpha(A)$
$\alpha_2(A) = c_\alpha(A)$ (α-closure)	$\alpha_9(A) = c. c_\alpha . c. c_\alpha . c(A)$
$\alpha_3(A) = c_\alpha . c(A)$	$\alpha_{10}(A) = c_\alpha . c. c_\alpha . c. (A)$

$\alpha_4(A) = c.c_\alpha(A)$	$\alpha_{11}(A) = c_\alpha.c.c_\alpha.c.c_\alpha.c(A)$
$\alpha_5(A) = c.c_\alpha.c(A)$ $(\alpha - interior)$	$\alpha_{12}(A) = c.c_\alpha.c.c_\alpha.c_\alpha(A)$
$\alpha_6(A) = c_\alpha.c.c_\alpha(A)$	$\alpha_{13}(A) = c.c_\alpha.c.c_\alpha.c.c_\alpha.c(A)$

The composition table for c_α is :

$\circ$	α_0	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}	α_{11}	α_{12}	α_{13}
α_0	α_0	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}	α_{11}	α_{12}	α_{13}
α_1	α_1	α_0	α_4	α_5	α_2	α_3	α_8	α_9	α_6	α_7	α_{12}	α_{13}	α_{10}	α_{11}
α_2	α_2	α_3	α_2	α_3	α_6	α_7	α_6	α_7	α_{10}	α_{11}	α_{10}	α_{11}	α_6	α_7
α_3	α_3	α_2	α_6	α_7	α_2	α_3	α_{10}	α_{11}	α_6	α_7	α_6	α_7	α_{10}	α_{11}
α_4	α_4	α_5	α_4	α_5	α_8	α_9	α_8	α_9	α_{12}	α_{13}	α_{12}	α_{13}	α_8	α_9
α_5	α_5	α_4	α_8	α_9	α_4	α_5	α_{12}	α_{13}	α_8	α_9	α_8	α_9	α_{12}	α_{13}
α_6	α_6	α_7	α_6	α_7	α_{10}	α_{11}	α_{10}	α_{11}	α_6	α_7	α_6	α_7	α_{10}	α_{11}
α_7	α_7	α_6	α_{10}	α_{11}	α_6	α_7	α_6	α_7	α_{10}	α_{11}	α_{10}	α_{11}	α_6	α_7
α_8	α_8	α_9	α_8	α_9	α_{12}	α_{13}	α_{12}	α_{13}	α_8	α_9	α_8	α_9	α_{12}	α_{13}
α_9	α_9	α_8	α_{12}	α_{13}	α_8	α_9	α_6	α_9	α_{12}	α_{13}	α_{12}	α_{13}	α_8	α_9
α_{10}	α_{10}	α_{11}	α_{10}	α_{11}	α_6	α_7	α_{10}	α_7	α_{10}	α_{11}	α_{10}	α_{11}	α_6	α_7
α_{11}	α_{11}	α_{10}	α_6	α_7	α_{10}	α_{11}	α_9	α_{11}	α_6	α_7	α_6	α_7	α_{10}	α_{11}
α_{12}	α_{12}	α_{13}	α_{12}	α_{13}	α_8	α_9	α_8	α_9	α_{12}	α_{13}	α_{12}	α_{13}	α_8	α_9
α_{13}	α_{13}	α_{12}	α_8	α_9	α_{12}	α_{13}	α_{12}	α_{13}	α_8	α_9	α_8	α_9	α_{12}	α_{13}

From the above table it is clear that $k(c_\alpha) = 14$.
In case of c_σ, the notations used for different 10 sets are :

$\sigma_0(A) = A$ $\quad$ $\sigma_4(A) = c.c_\sigma(A)$ $\quad$ $\sigma_8(A) = c_\sigma.c.c_\sigma.c(A)$

$\sigma_1(A) = c(A)$ $\quad$ $\sigma_5(A) = c.c_\sigma.c(A)$ $\quad$ $\sigma_9(A) = c.c_\sigma.c.c_\sigma.c(A)$

$\sigma_2(A) = c_\sigma(A)$ $\quad$ $\sigma_6(A) = c_\sigma.c.c_\sigma(A)$

$\sigma_3(A) = c_\sigma.c(A)$ $\quad$ $\sigma_7(A) = c.c_\sigma.c.c_\sigma(A)$

Like the above three cases, reader can verify that $k(c_\sigma) = 10$.

Below, we provide an example of a topological space for which all these bounds are realized. The elobrate details of the example is available in [6] and hence avoided here.

Example 2.3 Let$X = \mathbf{R}$, the set of real numbers, equipped with the usual topology. Then a subset of $A \subseteq X$ be defined by:

$A = \left\{-\frac{1}{n}, n \in \mathbf{N}\right\} \cup [[1,3] \backslash \{2 + \frac{1}{n}, n \in \mathbf{N}\}] \cup [(5,7] \cap \left(\mathbf{Q} \cup \bigcup_{n=1\ to\ \infty}(6 + \frac{1}{2n\pi}, 6 + \frac{1}{(2n-1)\pi}])\right] \cup (-3,-2]$. For this set, all the sets defined above are different.

If all the above four generalized closure operators and complement operator are taken together, they generate at the most **52**distinct sets under composition of operators. The reader may refer to [6] for further discussion in this regard. Similarly, the sets which satisfiy the property $A = i_\sigma c_\sigma(A)$ have been studiedn in [5] as PS-regular sets. In tour above discussion, a set is PS–regular if $A = \sigma_7$(A).

3. MONOIDS OF THE GENERAIZED CLOSURE OPERATORS

Taking composition of operators as the binary operation, we obtain the following monoids of operators:

$$\mathcal{M}_\pi = \{\pi_0, \pi_1, \ldots\ldots\ldots, \pi_8, \pi_9\}$$
$$\mathcal{M}_\alpha = \{\alpha_0, \alpha, \ldots\ldots\ldots, \alpha_{12}, \alpha_{13}\}$$
$$\mathcal{M}_\beta = \{\beta_0, \beta_1, \ldots\ldots\ldots, \beta_6, \beta_7\}$$
$$\mathcal{M}_\sigma = \{\sigma_0, \sigma_1, \ldots\ldots\ldots, \sigma_8, \sigma_9\}$$

This is clear from composition tables provided in the previous section.

Therorem 3.1 The generators of the monoids $\mathcal{M}_\delta, where\ \delta = \pi, \beta, \alpha\ and\ \sigma$ are given by
$i.)$ $\mathcal{M}_\delta = <\delta_1, \delta_i>$, *where* $i = 2,3,4,5$.

Proof. It follows from the fact that $\mathcal{M}_\pi = <\pi_1, \pi_2>$ and $\pi_2 = \pi_3 \circ \pi_1 = \pi_1 \circ \pi_4 = \pi_1 \circ \pi_5 \circ \pi_1$, to be verified from the composition table.
Same argument is valid for $\mathcal{M}_\alpha, \mathcal{M}_\beta$and$\mathcal{M}_\sigma$.

Theorem 3.2*The total number of distinct semi-groups generated by the members of $\mathcal{M}_\alpha$ is 118, under the composition of operators.*

Proof. We enlist all the semi groups contained in $\mathcal{M}_\alpha$ in the following manner. Since the calculation part may be easily verified from the composition table, we leave it to the reader.

i.) Semi-groups with one generator and one element:
$<\alpha_0> = \{\alpha_0\}$, $<\alpha_2> = \{\alpha_2\}$,$<\alpha_5> = \{\alpha_5\}$
$<\alpha_5> = \{\alpha_5\}$, $<\alpha_8> = \{\alpha_8\}$, $<\alpha_{10}> = \{\alpha_{10}\}$,
$<\alpha_{13}> = \{\alpha_{13}\}$

ii.) Semi groups with one generator and two elements.
$<\alpha_1> = \{\alpha_1, \alpha_0\}$, $<\alpha_6> = \{\alpha_6, \alpha_{10}\}$,

$<\alpha_{11}> = \{\alpha_7, \alpha_{11}\}$, $<\alpha_9> = \{\alpha_9, \alpha_{13}\}$,

$<\alpha_{12}> = \{\alpha_8, \alpha_{12}\}$, $<\alpha_{11}> = \{\alpha_7, \alpha_{11}\}$

iii.) Semi groups with one generator and more than two elements:

$$< \alpha_3 > = \{\alpha_3, \alpha_7, \alpha_{11}\}, < \alpha_4 > = \{\alpha_4, \alpha_8, \alpha_{12}\}$$

iii.) Semi groups with one generator and more than two elements:

Now, we provide an exhaustive list of all semi groups with two generators consisting of element of $\mathcal{M}_\alpha$.

$$< \alpha_1, \alpha_2 > = \{\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \dots \dots \dots, \alpha_{13}\}$$
$$< \alpha_1, \alpha_6 > = \{\alpha_0, \alpha_1, \alpha_6, \alpha_7, \alpha_8, \alpha_9, \dots, \alpha_{13}\},$$
$$<\alpha_3, \alpha_4> = \{\alpha_2, \alpha_3, \dots \dots \dots \dots, \alpha_{12}, \alpha_{13}\}$$
$$< \alpha_6, \alpha_9 > = \{\alpha_6, \alpha_7, \dots \dots \dots, \alpha_{12}, \alpha_{13}\}$$

It may be observed that

$$< \alpha_1, \alpha_2 > \supseteq < \alpha_1, \alpha_6 > \supseteq < \alpha_6, \alpha_9 >$$

and $< \alpha_1, \alpha_2 > \supseteq < \alpha_3, \alpha_4 > \supseteq < \alpha_6, \alpha_9 >$.

Further $< \alpha_6, \alpha_9 >$has 9 semi groups each having two generators. They are:

$$< \alpha_7, \alpha_{10} > = \{\alpha_7, \alpha_{10}\}, < \alpha_7, \alpha_{13} > = \{\alpha_7, \alpha_{13}\},$$
$$< \alpha_8, \alpha_{10} > = \{\alpha_8, \alpha_{10}\}, <\alpha_8, \alpha_{13}> = \{\alpha_8, \alpha_{13}\},$$
$$<\alpha_6, \alpha_7> = \{\alpha_6, \alpha_7, \alpha_{10}, \alpha_{11}\},$$
$$<\alpha_8, \alpha_9> = \{\alpha_8, \alpha_9, \alpha_{12}, \alpha_{13}\}$$
$$< \alpha_6, \alpha_8 > = \{\alpha_6, \alpha_8, \alpha_{10}, \alpha_{12}\},$$
$$< \alpha_7, \alpha_9 > = \{\alpha_7, \alpha_9, \alpha_{11}, \alpha_{13}\},$$
$$< \alpha_7, \alpha_8 > = \{\alpha_7, \alpha_8, \alpha_{10}, \alpha_{13}\}$$

Again $< \alpha_6, \alpha_9 >$ contains 8 semi groups with one generator, namely, $< \alpha_6 >, < \alpha_7 >, < \alpha_8 >, < \alpha_9 >, < \alpha_{10} >, < \alpha_{11} >, < \alpha_{12} >, < \alpha_{13} >$, which are already mentioned in the beginning.

Thus total numbers of semi groups of $< \alpha_6, \alpha_9 >$ are 9+8+1=18 (including itself).

Now, semi groups of $< \alpha_3, \alpha_4 >$, which are not listed above, with two generators are 27 in numbers. They are:

$$< \alpha_2, \alpha_6 > = \{\alpha_2, \alpha_6, \alpha_{10}\}, < \alpha_2, \alpha_{10} > = \{\alpha_2, \alpha_{10}\},$$
$$< \alpha_5, \alpha_{13} > = \{\alpha_5, \alpha_{13}\},$$
$$< \alpha_5, \alpha_9 > = \{\alpha_5, \alpha_9, \alpha_{13}\}$$
$$< \alpha_2, \alpha_7 > = \{\alpha_2, \alpha_7, \alpha_{10}\}, < \alpha_2, \alpha_8 > = \{\alpha_2, \alpha_8, \alpha_{10}\},$$
$$< \alpha_5, \alpha_7 > = \{\alpha_5, \alpha_7, \alpha_{13}\}$$
$$< \alpha_5, \alpha_8 > = \{\alpha_5, \alpha_8, \alpha_{13}\}$$
$$< \alpha_2, \alpha_{11} > = \{\alpha_2, \alpha_6, \alpha_7, \alpha_{10}, \alpha_{11}\},$$
$$< \alpha_2, \alpha_{12} > = \{\alpha_2, \alpha_6, \alpha_8, \alpha_{10}, \alpha_{12}\},$$
$$< \alpha_2, \alpha_{13} > = \{\alpha_2, \alpha_7, \alpha_8, \alpha_{10}, \alpha_{13}\}$$
$$< \alpha_2, \alpha_9 > = \{\alpha_2, \alpha_6, \alpha_7, \dots \dots \dots, \alpha_{12}, \alpha_{13}\},$$
$$< \alpha_5, \alpha_{12} > = \{\alpha_5, \alpha_8, \alpha_9, \alpha_{12}, \alpha_{13}\},$$
$$< \alpha_5, \alpha_{11} > = \{\alpha_5, \alpha_7, \alpha_9, \alpha_{11}, \alpha_{13}\},$$
$$< \alpha_5, \alpha_{10} > = \{\alpha_5, \alpha_7, \alpha_8, \alpha_{10}, \alpha_{13}\}$$
$$< \alpha_5, \alpha_6 > = \{\alpha_5, \alpha_6, \alpha_7, \dots \dots, \alpha_{12}, \alpha_{13}\}$$
$$< \alpha_3, \alpha_6 > = \{\alpha_3, \alpha_6, \alpha_7, \alpha_{10}, \alpha_{11}\}$$
$$< \alpha_3, \alpha_9 > = \{\alpha_3, \alpha_7, \alpha_9, \alpha_{11}, \alpha_{13}\}$$
$$< \alpha_3, \alpha_8 > = \{\alpha_3, \alpha_6, \alpha_7, \dots \dots \dots \dots \dots, \alpha_{12}, \alpha_{13}\}$$
$$<\alpha_4, \alpha_9> = \{\alpha_4, \alpha_8, \alpha_9, \alpha_{12}, \alpha_{13}\}$$
$$< \alpha_4, \alpha_6 > = \{\alpha_4, \alpha_6, \alpha_8, \alpha_{10}, \alpha_{12}\}$$
$$< \alpha_4, \alpha_7 > = \{\alpha_4, \alpha_6, \alpha_7, \dots \dots, \alpha_{12}, \alpha_{13}\}$$
$$< \alpha_2, \alpha_3 > = \{\alpha_2, \alpha_3, \alpha_6, \alpha_7, \alpha_{10}, \alpha_{11}\}$$
$$< \alpha_4, \alpha_5 > = \{\alpha_4, \alpha_5, \alpha_8, \alpha_9, \alpha_{12}, \alpha_{13}\}$$
$$< \alpha_2, \alpha_4 > = \{\alpha_2, \alpha_4, \alpha_6, \alpha_8, \alpha_{10}, \alpha_{12}\}$$
$$< \alpha_3, \alpha_5 > \{\alpha_3, \alpha_5, \alpha_7, \alpha_9, \alpha_{11}, \alpha_{13}\}$$
$$< \alpha_2, \alpha_5 > = \{ \alpha_2, \alpha_5, \alpha_7, \alpha_8, \alpha_{10}, \alpha_{13}\}$$

Similarly, we get 7 semi groups of $< \alpha_3, \alpha_4 >$ having 3 generators. They are:

$$< \alpha_2, \alpha_3, \alpha_8 > = \{\alpha_2, \alpha_3, \alpha_6, \alpha_7, \dots \dots, \alpha_{12}, \alpha_{13}\}$$
$$< \alpha_4, \alpha_5, \alpha_6 > = \{\alpha_4, \alpha_5, \alpha_7, \alpha_8, \dots \dots, \alpha_{12}, \alpha_{13}\}$$
$$< \alpha_2, \alpha_4, \alpha_7 > = \{\alpha_2, \alpha_4, \alpha_6, \alpha_7, \dots \dots \dots, \alpha_{12}, \alpha_{13}\}$$
$$< \alpha_3, \alpha_5, \alpha_8 > = \{\alpha_3, \alpha_5, \alpha_6, \alpha_7, \dots \dots, \alpha_{12}, \alpha_{13}\}$$
$$< \alpha_2, \alpha_5, \alpha_6 > = \{\alpha_2, \alpha_5, \alpha_6, \alpha_7, \dots \dots \dots, \alpha_{12}, \alpha_{13}\}$$
$$< \alpha_2, \alpha_3, \alpha_5 > = \{\alpha_2, \alpha_3, \alpha_5, \alpha_6, \alpha_7, \dots \dots \dots, \alpha_{12}, \alpha_{13}\}$$
$$< \alpha_2, \alpha_4, \alpha_5 > = \{\alpha_2, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \dots \dots \dots, \alpha_{12}, \alpha_{13}\}$$

Thus, altogether, total number of distinct semi groups generated by the elements of $\mathcal{M}_\alpha$ without α_0 amounts to 57. One can check that it enumerates four monoids with α_0 and 57 semi groups without α_0.The exhaustive list is provided below:

$< \alpha_0 >, < \alpha_1 >, < \alpha_2 >, < \alpha_3 >, < \alpha_4 >, < \alpha_5 >, < \alpha_6 >,$
$< \alpha_7 >, < \alpha_8 >, < \alpha_9 >, < \alpha_{10} >, < \alpha_{11} >, < \alpha_{12} >,$
$< \alpha_{13} >, < \alpha_1, \alpha_2 >, < \alpha_1, \alpha_6 >, <\alpha_3, \alpha_4 >, < \alpha_6, \alpha_9 >, <$
$\alpha_7, \alpha_{10} >, < \alpha_7, \alpha_{13} >,$
$< \alpha_8, \alpha_{10} >, <\alpha_8, \alpha_{13}>, <\alpha_6, \alpha_7>, <\alpha_8, \alpha_9>,$
$< \alpha_6, \alpha_8 >, < \alpha_7, \alpha_9 >, < \alpha_7, \alpha_8 >, < \alpha_2, \alpha_6 >, < \alpha_2, \alpha_{10}$
$>, < \alpha_5, \alpha_{13} >, < \alpha_5, \alpha_9 >, < \alpha_2, \alpha_7 >,$
$< \alpha_2, \alpha_8 >, < \alpha_5, \alpha_7 >, < \alpha_5, \alpha_8 >,$
$< \alpha_2, \alpha_{11} >, < \alpha_2, \alpha_{12} >, < \alpha_2, \alpha_{13} >,$
$< \alpha_2, \alpha_9 >, < \alpha_5, \alpha_{12} >, < \alpha_5, \alpha_{11} >,$
$< \alpha_5, \alpha_{10} >, < \alpha_5, \alpha_6 >, < \alpha_3, \alpha_6 >,$
$< \alpha_3, \alpha_9 >, < \alpha_1, \alpha_2 >, < \alpha_1, \alpha_6 >,$
$< \alpha_6, \alpha_9 >, < \alpha_3, \alpha_4 >,$
$< \alpha_3, \alpha_8 >, < \alpha_4, \alpha_9 >, < \alpha_4, \alpha_6 >, < \alpha_4, \alpha_7 >, < \alpha_2, \alpha_3 >,$
$< \alpha_4, \alpha_5 >, < \alpha_2, \alpha_4 >, < \alpha_3, \alpha_5 >,$
$< \alpha_2, \alpha_5 >, < \alpha_2, \alpha_3, \alpha_8 >, < \alpha_4, \alpha_5, \alpha_6 >,$
$< \alpha_2, \alpha_4, \alpha_7 >, < \alpha_3, \alpha_5, \alpha_8 >, < \alpha_2, \alpha_5, \alpha_6$
$>< \alpha_2, \alpha_3, \alpha_5 >, < \alpha_2, \alpha_4, \alpha_5 >$

Since$\alpha_0$is an identity operator in monoid $\mathcal{M}_\alpha$, therefore by adding $\alpha_0$to each of the semi group not containing$\alpha_{0,}$we again get a semi group of $\mathcal{M}_\alpha$, therefore there are $4 + 57.2 = 118$semi groups in $\mathcal{M}_\alpha$.

Similarly, the above study can be carried out for $\mathcal{M}_\pi, \mathcal{M}_\beta$ and $\mathcal{M}_\sigma$ also.

The reader may refer to [11] for similar algebraic treatment of Kuratowski operator.

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